NEHRU COLLEGE OF ENGINEERING AND RESEARCH CENTRE
(NAAC Accredited)
(Approved by AICTE, Affiliated to APJ Abdul Kalam Technological University, Kerala)

DEPARTMENT OF MECHATRONICS ENGINEERING

COURSE MATERIALS

MA 202 PROBABILITY DISTRIBUTIONS, TRANSFORMS AND NUMERICAL METHODS

VISION OF THE INSTITUTION
To mould true citizens who are millennium leaders and catalysts of change through excellence in education.
MISSION OF THE INSTITUTION
NCERC is committed to transform itself into a center of excellence in Learning and Research in Engineering and Frontier Technology and to impart quality education to mould technically competent citizens with moral integrity, social commitment and ethical values. We intend to facilitate our students to assimilate the latest technological know-how and to imbibe discipline, culture and spiritually, and to mould them in to technological giants, dedicated research scientists and intellectual leaders of the country who can spread the beams of light and happiness among the poor and the underprivileged.

ABOUT DEPARTMENT

- Established in: 2013
- Course offered: B.Tech Mechatronics Engineering
- Approved by AICTE New Delhi and Accredited by NAAC
- Affiliated to the University of Dr. A P J Abdul Kalam Technological University.

DEPARTMENT VISION

To develop professionally ethical and socially responsible Mechatronics engineers to serve the humanity through quality professional education.

DEPARTMENT MISSION

1) The department is committed to impart the right blend of knowledge and quality education to create professionally ethical and socially responsible graduates.

2) The department is committed to impart the awareness to meet the current challenges in technology.

3) Establish state-of-the-art laboratories to promote practical knowledge of mechatronics to meet the needs of the society

PROGRAMME EDUCATIONAL OBJECTIVES

I. Graduates shall have the ability to work in multidisciplinary environment with good professional and commitment.

II. Graduates shall have the ability to solve the complex engineering problems by applying electrical, mechanical, electronics and computer knowledge and engage in lifelong learning in their profession.

III. Graduates shall have the ability to lead and contribute in a team with entrepreneur skills, professional, social and ethical responsibilities.

IV. Graduates shall have ability to acquire scientific and engineering fundamentals necessary for higher studies and research.

PROGRAM OUTCOME (PO'S)

Engineering Graduates will be able to:

PO 1. Engineering knowledge: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.

PO 2. Problem analysis: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.

PO 3. Design/development of solutions: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.

PO 4. Conduct investigations of complex problems: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

PO 5. Modern tool usage: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.

PO 6. The engineer and society: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.

PO 7. Environment and sustainability: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.

PO 8. Ethics: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.

PO 9. Individual and team work: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.

PO 10. Communication: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.

PO 11. Project management and finance: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

PO 12. Life-long learning: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

PROGRAM SPECIFIC OUTCOME(PSO'S)

PSO 1: Design and develop Mechatronics systems to solve the complex engineering problem by integrating electronics, mechanical and control systems.

PSO 2: Apply the engineering knowledge to conduct investigations of complex engineering problem related to instrumentation, control, automation, robotics and provide solutions.

COURSE OUTCOME

After the completion of the course students will be able to:

CO 1: To introduce the concept of discrete random variables, probability distributions, with practical application in various Engineering and sociallife situations.
CO 2: To introduce the concept of discrete random variables, probability distributions, with practical application in various Engineering and sociallife situations.
CO 3: To know Laplace and its transform which has wide application in all Engineering courses.
CO 4: To know Fourier and its transform which has wide application in all Engineering courses.
CO 5: To enable the students to solve various engineering problems using numerical methods.
CO 6: To enable the students to know Numerical solution of system of Equations. Numerical Integration, and ordinary differential equation of First order..

CO VS PO'S AND PSO'S MAPPING

CO	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12	PS01	PSO1
CO 1	3	3	3	3	2	1	-	-	1	2	-	2	1	1
CO 2	3	3	3	3	2	1	-	-	1	2	-	2	1	1
CO 3	3	3	3	3	2	1	-	-	-	2	-	2	1	1
CO 4	3	3	3	3	2	1	-	-	-	2	-	2	1	1
CO 5	3	3	3	3	2	1	-	-	-	2	-	2	1	1
co 6	3	3	3	3	2	1	-	-	-	2	-	2	1	1

Note: H-Highly correlated=3, M-Medium correlated=2, L-Less correlated=1

SYLLABUS

Course No.	Course Name	L-T-P - Credits	Year of Introduction
MA202	**Probability distributions, Transforms and Numerical Methods**	**3-1-0-4**	**2016**

Prerequisite: Nil

Course Objectives

- To introduce the concept of random variables, probability distributions, specific discrete and continuous distributions with practical application in various Engineering and social life situations.
- To know Laplace and Fourier transforms which has wide application in all Engineering courses.
- To enable the students to solve various engineering problems using numerical methods.

Syllabus

Discrete random variables and Discrete Probability Distribution.
Continuous Random variables and Continuous Probability Distribution.
Fourier transforms.
Laplace Transforms.
Numerical methods-solution of Algebraic and transcendental Equations, Interpolation.
Numerical solution of system of Equations. Numerical Integration, Numerical solution of ordinary differential equation of First order.

Expected outcome .

After the completion of the course student is expected to have concept of
(i) Discrete and continuous probability density functions and special probability distributions.

(ii) Laplace and Fourier transforms and apply them in their Engineering branch
(iii) numerical methods and their applications in solving Engineering problems.

Text Books:

1. Miller and Freund's "Probability and statistics for Engineers"-Pearson-Eighth Edition.
2. Erwin Kreyszig, "Advanced Engineering Mathematics", 10th edition, Wiley, 2015.

References:

1. V. Sundarapandian, "Probability, Statistics and Queuing theory", PHI Learning, 2009.
2. C. Ray Wylie and Louis C. Barrett, "Advanced Engineering Mathematics"-Sixth Edition.
3. Jay L. Devore, "Probability and Statistics for Engineering and Science"-Eight Edition.
4. Steven C. Chapra and Raymond P. Canale, "Numerical Methods for Engineers"-Sixth Edition-Mc Graw Hill.

Course Plan

Module	Contents	Hours	Sem. Exam Marks
I	**Discrete Probability Distributions.** (Relevant topics in section 4.1,4,2,4.4,4.6 Text1)		
	Discrete Random Variables, Probability distribution function, Cumulative distribution function.	2	
	Mean and Variance of Discrete Probability Distribution.	2	
	Binomial Distribution-Mean and variance.	2	
	Poisson Approximation to the Binomial Distribution. Poisson distribution-Mean and variance.	2	15%

II	**Continuous Probability Distributions.** (Relevant topics in section 5.1,5.2,5.5,5.7 Text1) Continuous Random Variable, Probability density function, Cumulative density function, Mean and variance. Normal Distribution, Mean and variance (without proof). Uniform Distribution.Mean and variance. Exponential Distribution, Mean and variance.	2 4 2 2	15%
	FIRST INTERNAL EXAMINATION		
III	**Fourier Integrals and transforms. (**Relevant topics in section 11.7, 11.8, 11.9 Text2) Fourier Integrals. Fourier integral theorem (without proof). Fourier Transform and inverse transform. Fourier Sine & Cosine Transform, inverse transform.	3 3 3	15%
IV	**Laplace transforms. (**Relevant topics in section 6.1,6.2,6.3,6.5,6.6 Text2**)** Laplace Transforms, linearity, first shifting Theorem. Transform of derivative and Integral, Inverse Laplace transform, Solution of ordinary differential equation using Laplace transform. Theorem (without proof). Differentiation and Integration of transforms.	3 4 2	15%
	SECOND INTERNAL EXAMINATION		
V	**Numerical Techniques**.(Relevant topics in section.19.1,19.2,19.3 Text2) Solution Of equations by Iteration, Newton- Raphson Method. Interpolation of Unequal intervals-Lagrange's Interpolation formula. Interpolation of Equal intervals-Newton's forward difference formula, Newton's Backward difference formula.	2 2 3	20%
VI	**Numerical Techniques**. (Relevant topics in section 19.5,20.1,20.3, 21.1 Text2) Solution to linear System- Gauss Elimination, Gauss Seidal Iteration Method. Numeric Integration-Trapezoidal Rule, Simpson's 1/3 Rule. Numerical solution of firstorder ODE-Euler method, Runge-Kutta Method (fourth order).	3 3 3	20%
	END SEMESTER EXAM		

QUESTION PAPER PATTERN:

Maximum Marks : 100 Exam Duration: 3 hoursThe question paper will consist of 3 parts.
Part A will have 3 questions of 15 marks each uniformly covering modules I and II. Eachquestion may have two sub questions.

Part B will have 3 questions of 15 marks each uniformly covering modules III and IV. Eachquestion may have two sub questions.

Part C will have 3 questions of 20 marks each uniformly covering modules V and VI. Eachquestion may have three sub questions.

Any two questions from each part have to be answered.

QUESTION BANK

<table>
<tr><th colspan="5">MODULE I</th></tr>
<tr><th>Q:NO:</th><th>QUESTIONS</th><th>CO</th><th>KL</th><th>PAGE NO:</th></tr>
<tr><td>1</td><td>A random variable X ha the following probability mass function
<table><tr><td>x</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td></tr><tr><td>P(X)</td><td>0</td><td>k</td><td>2k</td><td>2k</td><td>3k</td><td>K^2</td><td>$2K^2$</td><td>$7K^2+k$</td></tr></table>Find i) value of K ii) P(0<x<5) iii) p(x≥6)</td><td>CO1</td><td>K2</td><td>4</td></tr>
<tr><td>2</td><td>Show that for a Poisson distribution with parameter λ , mean= variance =λ.</td><td>CO1</td><td>K5</td><td>10</td></tr>
<tr><td>3</td><td>An Insurance company agent accepts policies of 5 men , all of identical age and good health. Probability that a man of this age</td><td>CO1</td><td>K2</td><td>12</td></tr>
</table>

	will be alive 30 years is 2/3.Find the probability that in 30 years i) all 5 men ii) at least one men will be alive .			
4	Given that f(x)= $\frac{k}{2^x}$ is a probability distribution of a random variable that can take on the values x= 0,1,2,,3,and 4, find k. find cumulative distribution function.	CO1	K3	15
5	Prove that binomial distribution with parameters n and p can be approximated to Poisson distribution when n is large and p is small with np=λ, a constant.	CO1	K3	16
6	During one stage in the manufacture of integrated circuit chips , a coating must be applied. If 70 % of chips receive a thick enough coating ,use Binomial distribution find the probabilities that among 15 chips, i) At least 12 will have thick enough coating ii) At most 6 will have thick enough coating iii) Exactly 10 will have thick enough coating	CO1	K2	18
7	If the sum of the mean and variance of a binomial distribution for 5 trials is 1.8. Find the probability distribution function	CO1	K5	20
8	A random variable x takes the values -3,-2,-1,0,1,2,3 such that P(X=0) =P(X>0)=P(X<0) and P(X=-3) =P(x=-2) =P(X=-1) =P(X=1)=P(X=2)=P(X=3). Obtain the probability distribution and the distribution function of X	CO1	K4	18
9	Derive the formula for mean and variance of binomial distribution	CO1	K2	26
10	The probability mass function of a random variable x is given below.	CO1	K5	14

x	0	1	2	3
P(X)	c	$2c^2$	C^2	$3c^2$

	Find i) Determine the value of C ii) $P(x \geq 1)$ iii) $p(x>1/0<x<3)$ iv) $E(X)$			
MODULE II				
1	The actual amount of instant coffee that a filling machine puts in to "4"ounce jars may belooked upon as a random variable having a normal distribution with σ = 0.04 ounce. If only 2%of the jars areto contain less than 4 ounces, what should be the mean fill of these jars.	CO2	K5	38
2	In a certain city, the number of power outages per month is a random variable having a distributionwith μ =11.6 and σ = 3.3. If this distribution can be approximated closely with a normal distribution. What is the probability that there will be at least 8 outages in any one month	CO2	K2	41
3	In a certain experiment the error made in determining the solubility of a substance is a random variable having uniform distribution with a = -0.025 and β = 0.025. What are the probabilities that such an error will be between a) 0.010 and 0.015 b) -0.012 and 0.012?	CO2	K2	57
4	Derive the mean and variance of Uniform Distribution	CO2	K5	41
5	If a random variable has the probability density $f(x) = 2e^{-2x}$ $\quad for\ x > 0$ $for\ x \leq 0$ Find the probabilities that it will take on a value. i) Between 1 and 3. ii) Greater than 0.5 iii) Find the mean and variance of X	CO2	K3	55

6	Find k such that the following can serve as the probability density of a random variable $F(x) = kxe^{-4x2}$,x>0	CO2	K5	40
7	In an intelligence test administered on 1000 children, the average was 60 and standard deviation was 20 Assuming that the marks obtained by the children follow normal distribution, find the number of children who have scored i) Over 90 marks ii)Below 40 marks iii) Between 50 and 80 marks.	CO2	K4	56
8	The number of personal computers sold daily at CompuWorld is uniformly distributed with minimum of 2000 PCs and maximum of 5000 PCs. What is the probability that (i) The daily sales will fall between 2500 and 3000 PCs (ii) CompuWorld will sell at least 4000 PCs (iii) CompuWorld will exactly sell 2500 PCs.	CO2	K3	47
9	The time (in hours) required to repair a machine is exponentially distributed with parameter λ =1 what is the probability that the repair time exceeds 3 hours?	CO2	K2	43
MODULE III				
1	1) Find the Fourier integral representation of the function $f(x) = \begin{cases} 1 & \text{if} \lvert x \rvert < 1 \\ 0 & \text{if} \lvert x \rvert < 1 \end{cases}$	CO3	K4	78
2	Derive the Fourier cosine and Fourier sine integral of $f(x) = e^{-kx}, x > 0, k > 0$	CO3	K3	84
3	Find the Fourier cosine and Fourier sine integral representation of $f(x)$=sinx , 0<x<π	CO3	K2	100
4	Prove the Linearity property of Fourier transform.	CO3	K2	92
5	Find the Fourier Transform of Unit Step function.	CO3	K5	94

6	Find the Fourier Transform of Unit impulse function.	CO3	K4	97
7	Use Fourier integral to show that $\int_0^\infty \frac{cosxw+wsinxw}{1+w^2}$ $=\begin{cases} 0 & if \quad x<0 \\ \frac{\pi}{2} & x=0 \\ \pi e^{-x} & x>0 \end{cases}$	CO3	K2	105
8	Find the Fourier transform of $f(x) = \ 1 \ \ if \ \ \lvert x\rvert < 1$ 0 otherwise	CO3	K3	102
9	Find the Fourier transform of e^{-x^2} .Hence find $F(xe^{-x^2})$	CO3	K3	105
10	Find the Fourier Sine Transform of $\frac{1}{x}$	CO3	K5	95
11	Express $f(x) = 1 \quad ,0 < x < \pi$ 0 , $x > \pi$ as a Fourier sine integral and evaluate $\int_0^\infty \frac{1-cos\pi w}{w} \ sinxw \ dw$	CO3	K2	100
	MODULE IV			
1	Find the Laplace transform of (i) sinht cost (ii) $3t^2 e^{-4t}$	CO4	K2	110
2	Find inverse Laplace Transform of (i) $\frac{1}{(s^2+1)(s^2+9)}$ (ii) $\frac{1}{s^2-4s+8}$ (iii) $\frac{\pi}{S^2+4S\pi+3\pi^2}$	CO4	K1	107

3	Find inverse Laplace Transform of (i) $\frac{s-3}{(s-3)^2+9}$ (ii) $\frac{4s+12}{s^2+8s+16}$ (iii) $\frac{S^2+2}{(S-1)^2(S^2+4)}$	CO4	K2	120
4	Using Laplace transform, Solve the IVP $y''-y'+9y=0$, $y(0)=0.16$, $y'(0)=0$	CO4	K3	115
5	Find Laplace transform of $(t-3)^2U(t-3)$	CO4	K1	130
6	Find Laplace transform $te^{2t}\sin 3t$	CO4	K2	123
7	Using Laplace transform, Solve the IVP $y''-y=t$, $y(0)=1$, $y'(0)=1$	CO4	K3	138
8	Find inverse Laplace Transform of $\frac{Se^{-3S}}{S^2-a^2}$	CO4	K3	109
9	Find the Laplace transform of (i) coshat coswt (ii) 5t sin2t	CO4	K2	152
	MODULE V			
1	Using Newton Raphson method compute the square root of 51 correct to 4 decimal places	CO5	K4	160
2	For the following data calculate the value of y when x= 9 X 8 10 12 14 16 18 Y 10 19 32.5 54 89.5 154	CO5	K2	170

3	Given f(2) = 5 , f(2.5)=6 find the linear interpolating polynomial using LAGRANGE ' S formula and also find f(2.2)	CO5	K3	184
4	Determine the interpolating polynomial for the following data X -1 0 1 3 Y 2 1 0 -1	CO5	K2	185
5	solve f(x)=x-0.5 cosx=0 near x=0 by fixed point Iteration method	CO5	K3	180
6	Find f(9.2) from the values given below by Lagrange' S interpolation X 8 9 9.5 f(x) 2.197225 2.251292 2.397895	CO5	K2	190
7	Given (xj , f(xj) = (0.2, 0.9980), (0.4, 0.9686) , (0.8 , 0.5358) , (1,0) find f(0.7) based on these values using Newton 's interpolation formula	CO5	K2	183
MODULE VI				
1	Solve $y' = -2x^3 + 12x^2 - 20x + 8.5$, $y(0) = 1$ $for\ x = 0.5$ using i) Eulermethod ii)Rumge-kutta method . Also find the error in two methods	CO6	K3	196
2	Solve by Gauss Seidal iteration method $3x + 2y + z = 7$, $x + 3y + 2z = 4$, $2x + y + 3z = 7$	CO6	K3	190
3	Using fourth Runge-kutta method solve the initial value problem $y' = x + y$, $y(0) = 1 in\ the\ intervel\ (0, 0.2) by\ taking\ h = 0.1$	CO6	K2	188
4	Evaluate log 7 by Simpson's rule(hint: log 7=$\int_0^6 \frac{dx}{1+x}$)	CO6	K5	184

5	Solve the differential equation $\frac{dy}{dx} = \frac{y-x}{y+x}$, y(0)=1 for x=.1, h=.02 using Euler's method 1	CO6	K2	199
6	Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule with n=10	CO6	K2	188
7	solve f(x)=x-0.5 cosx=0 near x=0 by fixed point Iteration method	Co6	K3	190
8	Solve the differential equation $\frac{dy}{dx} = \frac{y+x}{y-x}$, y(0)=1 for x=.1, h=.02 using Euler's method	CO6	K2	192

MODULE - I

DISCRETE PROBABILITY DISTRIBUTIONS

Random Experiment.

An experiment that can result in different outcomes eventhough it is repeated under same conditions every time is called Random experiment

Sample space

The set of all outcomes of a random experiment is called sample space 'S'.

Random variable (r.v)

A random variable is a variable x that assigns a real number for every outcome of a Random experiment.

Types of Random variable

~~A r.v is said to be~~

1. Discrete random variable

If a r.v takes only a finite number of values or countably infinite number of values then the r.v is called a discrete r.v.

Eg: a) Let x denote the number of heads obtained when two coins are tossed.

ie $X = 0, 1, 2$

b) In die throwing experiment, the r.v 'x' denote the number of points obtained.

ie $X = 1, 2, 3, 4, 5, 6$.

CONTINOUS RANDOM VARIABLES

A r.v is said to be continous r.v if x takes all possible values in an interval or infinite number of points.

Probability Mass function / Discrete probability distribution (PMF)

The function $f(x) = P[X = x]$ of discrete r.v is said to be a PMF if it satisfies the following conditions

i) $f(x) \geq 0$ and

ii) $\sum f(x) = 1$

Problems :-

1) check whether the following function can be the PMF of a r.v X.

a) $f(x) = x-3$, $x = 1, 2, 3, 4, 5$

Soln:

given,

X :	1	2	3	4	5
f(x) :	-2	-1	0	1	2

clearly $f(1) = -2$ and $f(2) = -1$ are negative.

$\therefore$ f(x) cannot be a PMF.

b) $f(x) = \frac{x^2}{30}$, $x = 0, 1, 2, 3, 4$

Soln:

X :	0	1	2	3	4
f(x) :	0	1/30	4/30	9/30	16/30

clearly all $f(x) \geq 0$ and

$$\Sigma f(x) = 0 + \frac{1}{30} + \frac{4}{30} + \frac{9}{30} + \frac{16}{30}$$

$$= 1$$

$\therefore$ f(x) is PMF.

Q) A rv x has the following PMF

x :	1	2	3	4	5	6	7
$f(x)$:	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find the following

i) value of k

ii) $P(0<x<5)$

iii) $P(x>5)$

Solution:

i) Given that $f(x)$ is PMF.

$\therefore \quad \Sigma f(x) = 1$

$\Rightarrow \quad k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow \quad 10k^2 + 9k - 1 = 0$

$\Rightarrow \quad k = \frac{1}{10}, -1.$

Since $f(x)$ always ≥ 0, $k=-1$ is rejected.

$\therefore$ Value of $k = \frac{1}{10}$

$\therefore$ PMF of x is

x :	1	2	3	4	5	6	7
$f(x)$:	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{7}{100}+\frac{1}{10}$

ii) $P(0<x<5) = P(x = 1, 2, 3, 4)$

$= P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$= \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10}$

$= \frac{8}{10}$

iii) $P(x>5)$

$= P(x=6,7)$

$= P(x=6) + P(x=7)$

$= \frac{2}{100} + \left(\frac{7}{100}+\frac{1}{10}\right)$

$= \frac{19}{100}$

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2) The PMF of a r.v X is

X :	0	1	2	3	4	5	6
p(x)	k	3k	5k	7k	9k	11k	13k

i) Find k ?

ii) $P(X \le 4)$

iii) $P(3 \le X \le 6)$

Soln:

i) Since given p(x) is PMF,

$\sum p(x) = 1$

$\Rightarrow k + 3k + 5k + 7k + 9k + 11k + 13k = 1$

$\Rightarrow \boxed{k = \frac{1}{49}}$

X :	0	1	2	3	4	5	6
p(x) :	$\frac{1}{49}$	$\frac{3}{49}$	$\frac{5}{49}$	$\frac{7}{49}$	$\frac{9}{49}$	$\frac{11}{49}$	$\frac{13}{49}$

ii) $P(X \le 4)$

$= 1 - P(X>4)$

$= 1 - [P(X=5,6)]$

$= 1 - [P(X=5) + P(X=6)]$

$= 1 - \left[\frac{11}{49} + \frac{13}{49}\right] = 1 - \frac{24}{49}$

$= 25/49$ //

iii) $P(3 \le x \le 6)$

$= P(x = 3, 4, 5, 6)$

$= P(x=3) + P(x=4) + P(x=5) + P(x=6)$

$= \frac{7}{49} + \frac{9}{49} + \frac{11}{49} + \frac{13}{49}$

$= \frac{40}{49}$

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2) If the probability distribution of a discrete r.v x is given as,

$$P(X=x) = \begin{cases} \frac{x}{15}, & x = 1, 2, 3, 4, 5 \\ 0, & \text{otherwise.} \end{cases}$$

Find i) $P(1 \text{ or } 2)$

ii) $P(1/2 < x < 5/2 \,/\, x > 1)$

Solution:

given that,

x	1	2	3	4	5
$P(X=x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

i) $P(1 \text{ or } 2)$

$= P(x=1) + P(x=2)$

$= \frac{1}{15} + \frac{2}{15}$

$= \frac{3}{15}$

ii) $P\left(\frac{1}{2} < x < \frac{5}{2} \,/\, x > 1\right)$

$$= \frac{P\left[\left(\frac{1}{2} < x < \frac{5}{2}\right) \cap x > 1\right]}{P(x>1)}$$

$$= \frac{P\left[1 < x < \frac{5}{2}\right]}{P[x>1]}$$

$$= \frac{P[x=2]}{P(x=2,3,4,5)}$$

$$= \frac{2/15}{\frac{2}{15} + \frac{3}{15} + \frac{4}{15} + \frac{5}{15}}$$

$$= \frac{1}{7}$$

DISTRIBUTION FUNCTION (Cumulative Distribution function CDF or D.F)

$$\boxed{F(x) = P(X \le x)}$$

Q) $f(x) = \frac{k}{2^x}$, is probability distribution of rv X that can takes values 0, 1, 2, 3 and 4

i) Find k

ii) Find distribution function (D.F)?

Solution:-

Given $f(x)$ is PMF,

$\Sigma f(x) = 1$

x:	0	1	2	3	4
$f(x)$:	$\frac{k}{2^0}$	$\frac{k}{2^1}$	$\frac{k}{2^2}$	$\frac{k}{2^3}$	$\frac{k}{2^4}$

$\Rightarrow$

$$\Rightarrow \quad k + \frac{k}{2} + \frac{k}{4} + \frac{k}{8} + \frac{k}{32} = 1$$

$$\Rightarrow \quad \boxed{k = \frac{16}{31}}$$

$\therefore$ PMF of X is

X:	0	1	2	3	4
$f(x)$:	$\frac{16}{31}$	$\frac{8}{31}$	$\frac{4}{31}$	$\frac{2}{31}$	$\frac{1}{31}$

ii) Distribution function of $f(x)$ is,

$$F(x) = P(X \leq x)$$

when $x=0$, $F(0) = P(X \leq 0)$

$$= P(X=0)$$

$$= \frac{16}{31}$$

when $x=1$, $F(1) = P(X \leq 1)$

$$= P(X=0,1)$$

$$= P(X=0) + P(X=1)$$

$$= \frac{16}{31} + \frac{8}{31}$$

$$= \frac{24}{31}$$

when $x=2$, $F(2) = P(X \leq 2)$

$$= P(X=0,1,2)$$

$$= P(X=0) + P(X=1) + P(X=2)$$

$$= \frac{16}{31} + \frac{8}{31} + \frac{4}{31}$$

$$= \frac{28}{31}$$

when $x = 3$, $F(3) = P(x \leq 3)$

$$= P(x=0) + P(x=1) + P(x=2) + P(x=3)$$

$$= \frac{16}{31} + \frac{8}{31} + \frac{4}{31} + \frac{2}{31}$$

$$= \frac{30}{31}$$

when $x = 4$, $F(4) = P(x \leq 4)$

$$= P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4)$$

$$= \frac{16}{31} + \frac{8}{31} + \frac{4}{31} + \frac{2}{31} + \frac{1}{31}$$

$$= 1$$

$\therefore$ CDF or D.F of $f(x)$ is

x :	0	1	2	3	4
$F(x)$:	$\frac{16}{31}$	$\frac{24}{31}$	$\frac{28}{31}$	$\frac{30}{31}$	1

H.W

EXPECTATION OF A DISCRETE R.V

MEAN AND VARIANCE.

Let x be a discrete r.v. Then Expectation or Mean of x is denoted by $E(x)$.

ie Expectation = Mean = $E(x) = \Sigma x \cdot p(x)$

$$\Rightarrow \quad E(x) = \Sigma x \cdot p[x = x]$$

Also Variance of r.v x is denoted by $V(x)$.

ie Variance of x = $V(x) = E(x^2) - [E(x)]^2$

where $E(x) = \Sigma x \cdot p(x)$

$E(x^2) = \Sigma x^2 \cdot p(x)$

Standared Deviation $(SD) = \sqrt{V(x)}$.

Problems:-

Q) Let x be a random variable with PMF

x :	4	5	6	7	8	9
f(x) :	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Find Mean, variance and standared deviation of x. ?

Solution

$$\text{Mean} = E(x) = \Sigma x \cdot p(x) \text{ or } \Sigma x \cdot f(x)$$

$$= 4 \cdot \frac{1}{12} + 5 \cdot \frac{1}{12} + 6 \cdot \frac{1}{4} + 7 \cdot \frac{1}{4} + 8 \cdot \frac{1}{6} + 9 \cdot \frac{1}{6}$$

$$= \frac{9}{12} + \frac{13}{4} + \frac{17}{6} = \frac{41}{6}$$

$$\therefore \boxed{E(x) = \text{Mean} = \frac{41}{6}}$$

Now $E(x^2) = \Sigma x^2 p(x)$ or $\Sigma x \cdot f(x)$

$$= 4^2 \cdot \frac{1}{12} + 5^2 \cdot \frac{1}{12} + 6^2 \cdot \frac{1}{4} + 7^2 \cdot \frac{1}{4} + 8^2 \cdot \frac{1}{6} + 9^2 \cdot \frac{1}{6}$$

$$\boxed{E(x^2) = \frac{293}{6}}$$

$$\therefore \text{Variance} = V(x) = E(x^2) - [E(x)]^2$$

$$= \frac{293}{6} - \left(\frac{41}{6}\right)^2$$

$$\Rightarrow \boxed{V(x) = 2.1}$$

$$\therefore S.D = \sqrt{V(x)}$$

$$= \sqrt{2.1}$$

Q) A random variable x follows a PMF as given below.

x	0	1	2	3
$f(x)$	$\frac{k}{2}$	$\frac{k}{3}$	$\frac{k+1}{3}$	$\frac{2k+1}{6}$

Find i) k ii) Mean iii) Variance of x.

Given that $f(x)$ is PMF.

i) $\therefore \quad \Sigma f(x) = 1$

$$\Rightarrow \frac{k}{2} + \frac{k}{3} + \frac{k+1}{3} + \frac{2k+1}{6} = 1$$

$$\Rightarrow \frac{k}{2} + \left(\frac{2k+1}{3}\right) + \frac{2k+1}{6} = 1$$

$$\Rightarrow \frac{7k+2}{6} + \frac{2k+1}{6} = 1$$

$$\Rightarrow 9k+3 = 6$$

$$\Rightarrow k = \frac{3}{9}$$

$$\Rightarrow \boxed{k = \frac{1}{3}}$$

$\therefore$ PMF of x is

x :	0	1	2	3
$f(x)$:	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{5}{18}$

ii) Mean of x

$$\Rightarrow E(x) = \Sigma x \cdot f(x)$$

$$= 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{9} + 2 \cdot \frac{4}{9} + 3 \cdot \frac{5}{18}$$

$$\boxed{E(x) = \frac{11}{6}}$$

Now $E(x^2) = \Sigma x^2 . f(x)$

$$= 0^2 \cdot \frac{1}{6} + 1^2 \cdot \frac{1}{9} + 2^2 \cdot \frac{4}{9} + 3^2 \cdot \frac{5}{18}$$

$$= \frac{79}{18}$$

$\therefore$ Variance, $V(x) = E(x^2) - [E(x)]^2$

$$= \frac{79}{18} - \left(\frac{11}{6}\right)^2 = \underline{\underline{1.16}}$$

$$\boxed{V(X) = 1.16}$$

Q) Two unbiased dice are thrown. Find the expected value of the sum of the numbers thrown.

Soln:

Two dice are thrown.

$$\therefore S = \begin{Bmatrix} (1,1), (1,2) \ldots (1,6) \\ (2,1), (2,2) \ldots (2,6) \\ \vdots \\ (6,1), (6,2) \ldots (6,6) \end{Bmatrix}$$

Total no $= 36$

Let x denote the sum of numbers obtained.

ie r.v x takes values,

$x = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.$

PMF of x is,

x :	2	3	4	5	6	7	8	9	10	11	12
$p(x)$:	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Expected value of x = $E(X)$ = Mean = $\sum x \cdot p(x)$

$$= 2\cdot\frac{1}{36} + 3\cdot\frac{2}{36} + 4\cdot\frac{3}{36} + 5\cdot\frac{4}{36} + 6\cdot\frac{5}{36} + 7\cdot\frac{6}{36} + 8\cdot\frac{5}{36} + 9\cdot\frac{4}{36} +$$

$$+ 10\cdot\frac{3}{36} + 11\cdot\frac{2}{36} + 12\cdot\frac{1}{36}$$

$\Rightarrow E(x) = \frac{252}{36} = 7$

$\Rightarrow \boxed{E(x) = 7}$

---

2) The probability mass function of a discrete random variable is $p(x) = kx$, $x = 1, 2, 3$ where k is a positive constant. Find the

i) value of k

ii) $P(X \leq 2)$

iii) $E(X)$

iv) $Var(1 - X)$

Soln:

Given $p(x) = kx$, $x = 1, 2, 3$

X :	1	2	3
$p(x)$:	k	$2k$	$3k$

i) since $p(x)$ is PMF, $\Sigma p(x) = 1$

$k + 2k + 3k = 1$

$\Rightarrow 6k = 1$

$\Rightarrow \boxed{k = \frac{1}{6}}$

$\therefore$

X :	1	2	3
$p(x)$:	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

ii) $P(X \le 2)$

$= P(X = 1, 2)$

$= P(X=1) + P(X=2)$

$= \frac{1}{6} + \frac{2}{6} = \frac{3}{6} = \frac{1}{2}$

iii) ~~Var(1−x)~~ $E(x)$

$= \cancel{Var(1)} + \cancel{(-1)^2 V(x)}$

$= \cancel{0 + 1}.$

$\Rightarrow E(x) = \Sigma x \cdot p(x)$

$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{3}{6} = \frac{12}{6} = 2$

$\Rightarrow \boxed{E(x) = 2}$

Now $E(x^2) = \Sigma x^2 \cdot p(x)$

$= 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{2}{6} + 3^2 \cdot \frac{3}{6} = \frac{36}{6} = 6$

$\Rightarrow \boxed{E(x^2) = 6}$

$\therefore V(x) = E(x^2) - [E(x)]^2$

$= 6 - 2^2$

$= 2$

$\Rightarrow \boxed{V(x) = 2}$

iv) $Var(1-x) = Var(1) + (-1)^2 V(x)$

$= 0 + 1 \cdot 2$

$= 2$

$\Rightarrow \boxed{Var(1-x) = 2}$

Q) Let x denote the number that shows up when an unfair die is tossed. Faces 1 to 5 of the die are equally likely, while face 6 is twice as likely as any other. Find the probability distribution, mean and variance of x. ?

Soln:

Given faces 1 to 5 of die are equally likely.

$\therefore \quad P(X=1) = P(X=2) = P(X=3) = P(X=4) = P(X=5) = \frac{1}{6}$

but face 6 is twice as likely as any other.

$\therefore \quad P(\cancel{X}=6) = P(6) = \frac{2}{6}$

But that doesn't work because that would sum to $\frac{7}{6} \neq 1$.

So we choose 7 as our sample space

ie

X	1	2	3	4	5	6
P(x)	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{2}{7}$

is PMF of X.

Mean $= E(x) = \sum x\, p(x)$

$= 1\cdot\frac{1}{7} + 2\cdot\frac{1}{7} + 3\cdot\frac{1}{7} + 4\cdot\frac{1}{7} + 5\cdot\frac{1}{7} + 6\cdot\frac{1}{7}$

$\boxed{E(x) = \frac{21}{7}} = 3$

$E(x^2) = \sum x^2 \cdot p(x)$

$= 1^2\cdot\frac{1}{7} + 2^2\cdot\frac{1}{7} + 3^2\cdot\frac{1}{7} + 4^2\cdot\frac{1}{7} + 5^2\cdot\frac{1}{7} + 6^2\cdot\frac{2}{7}$

$E(x^2) = \frac{127}{7}$

$$\therefore \quad V(x) = E(x^2) - [E(x)]^2$$

$$= \frac{127}{7} - 3^2$$

$$V(x) = \frac{64}{7}$$

Q) ~~An or~~ A random variable x takes values 1, 2, 3, 4 such that, $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$. Find PMF and CDF of X ?

Soln:

Given $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = k$

clearly, $2P(X=1) = k$

$\Rightarrow P(X=1) = \frac{k}{2}$

$3P(X=2) = k \Rightarrow P(X=2) = \frac{k}{3}$

$P(X=3) = k$

$5P(X=4) = k \Rightarrow P(X=4) = \frac{k}{5}$

$\therefore$ PMF of X is

X :	1	2	3	4
$P(x)$:	$\frac{k}{2}$	$\frac{k}{3}$	k	$\frac{k}{5}$

$\Sigma P(X) = 1 \Rightarrow \frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$

$\Rightarrow \boxed{k = \frac{30}{61}}$

~~How are a Cumulative distribution functions~~,

∴ PMF of x is,

x	1	2	3	4
p(x)	$\frac{30}{122}$	$\frac{30}{183}$	$\frac{30}{61}$	$\frac{30}{305}$

Now Cumulative distribution function CDF is,

$$F(x) = P(X \leq x)$$

$$F(1) = P(X \leq 1) = P(X=1) = \frac{30}{122} = \frac{15}{61}$$

$$F(2) = P(X \leq 2) = P(X=1) + P(X=2) = \frac{30}{122} + \frac{30}{183} = \frac{25}{61}$$

$$F(3) = P(X \leq 3) = P(X=1) + P(X=2) + P(X=3) = \frac{15}{61} + \frac{25}{61} + \frac{30}{61} = \frac{70}{61}$$

$$F(4) = P(X \leq 4) = P(X=1) + P(X=2) + P(X=3) + P(X=4) = \underline{\underline{1}}$$

∴ CDF of x is

X	1	2	3	4
F(x)	$\frac{15}{61}$	$\frac{25}{61}$	$\frac{70}{61}$	1

SPECIAL DISCRETE PROBABILITY DISTRIBUTION

1 BINOMIAL DISTRIBUTION (B.D)

A Binomial random variable x is a discrete r.v having probability distribution,

$$P(X=x) = nc_x \, p^x q^{n-x}, \quad x = 0, 1, 2 \cdots n$$

where $0<p<1$ and $p+q=1$, ie $q = 1-p$

Then we say that x follows Binomial distribution

$x \rightarrow$ number of success.

$n \rightarrow$ number of trials.

$p \rightarrow$ probability of success.

$q \rightarrow$ probability of failure.

$n-x \rightarrow$ number of failures.

It is also denoted by $x \sim B(n, p)$.

'n' and 'p' are called parameters of B.D.

MEAN AND VARIANCE OF BINOMIAL DISTRIBUTION

Mean:-

$$E(x) = \sum_x x \cdot p(x) \quad \text{where } P(x) = nc_x \, p^x q^{n-x}$$

$$= \sum x \cdot nc_x \, p^x q^{n-x}$$

$$= \sum x \cdot \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$\boxed{nc_x = \frac{n!}{x!(n-x)!}}$$

$$= \sum x \cdot \frac{n(n-1)!}{x(x-1)!(n-x)!} p^x q^{n-x}$$

$$\boxed{n! = n(n-1)!}$$

$$\Rightarrow E(x) = n \sum_{x=1}^{n-1} \frac{(n-1)!}{(x-1)!(n-x)!} p \cdot p^{x-1} \cdot q^{n-x}$$

$$= np \sum_{x=1}^{n-1} \frac{(n-1)!}{(x-1)![(n-1)-(x-1)]!} p^{x-1} q^{(n-1)-(x-1)}$$

$$= np \left[(q+p)^{n-1} \right]$$ using binomial thm.

$$= np [1]$$ $(q+p)^{n-1} = \underline{\underline{1}}$

$$\Rightarrow \boxed{E(x) = np}$$

Now VARIANCE:-

$$V(x) = E(x^2) - [E(x)]^2$$

$$E(x^2) = \sum x^2 . P(x)$$

$$= \sum_{x=0}^{n} x^2 . {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} [x(x-1)+x] . \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) . \frac{n!}{x!(n-x)!} p^x q^{n-x} + \sum_{x=0}^{n} x . \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{x!(n-x)!} p^x q^{n-x} + np$$

$\therefore E(x) = np$

$$\Rightarrow E(x^2) = n(n-1)\sum_{x=2}^{n-2} \frac{x(x-1)(n-2)!}{x(x-1)(x-2)!(n-x)!} p^2 \cdot p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n-2} \frac{(n-2)!}{(x-2)![n-2-(x-2)]!} p^{x-2} q^{[n-2-(x-2)]} + np$$

$$= n(n-1)p^2 \left[(q+p)^{n-2}\right] + np$$

$$\boxed{\therefore (q+p)^{n-2} = 1}$$

$$\boxed{E(x^2) = n(n-1)p^2 + np}$$

$$\therefore\ V(x) = E(x^2) - [E(x)]^2$$

$$= \left(n(n-1)p^2 + np\right) - (np)^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$= npq \qquad \therefore q = 1-p,\ p+q=1$$

$$\therefore \quad \boxed{V(x) = npq}$$

Mean of B.D $= np$
Variance of B.D $= npq$

problems :-

Q) Ten coins are thrown simultaniously. Find the probability of getting atleast 7 heads.

Soln:

Let the r.v x represent the number of heads.

Then $x \sim B(n, p)$

p = probability of getting heads. $= \frac{1}{2}$

ie $p = \frac{1}{2}$ $\therefore$ $q = 1-p = 1-\frac{1}{2} = \frac{1}{2}$

and $n = 10$

The probability distribution of x is given by,

$$p(x=x) = nc_x \, p^x q^{n-x}, \quad x = 0,1,2 \dots n$$

ie $$p(x=x) = 10c_x \left(\tfrac{1}{2}\right)^x \left(\tfrac{1}{2}\right)^{10-x}, \quad x = 0,1,2 \dots 10$$

probability of getting atleast 7 heads is,

$$\Rightarrow p(x \geq 7) = p(x=7) + p(x=8) + p(x=9) + p(x=10)$$

$$= 10c_7 \left(\tfrac{1}{2}\right)^7 \left(\tfrac{1}{2}\right)^{10-7} + 10c_8 \left(\tfrac{1}{2}\right)^8 \left(\tfrac{1}{2}\right)^{10-8} + 10c_9 \left(\tfrac{1}{2}\right)^9 \left(\tfrac{1}{2}\right)^{10-9} + 10c_{10} \left(\tfrac{1}{2}\right)^{10} \left(\tfrac{1}{2}\right)^{10-10}$$

$$= 10c_7 \left(\tfrac{1}{2}\right)^{10} + 10c_8 \left(\tfrac{1}{2}\right)^{10} + 10c_9 \left(\tfrac{1}{2}\right)^{10} + 10c_{10} \left(\tfrac{1}{2}\right)^{10}$$

$$\therefore p(x \geq 7) = \underline{\underline{0.171875}}$$

Q) The mean and variance of a binomial r.v X are 16 and 8 respectively. Find $P(X=0)$ and $P(X=1)$

Soln:

$$\text{Mean} = E(X) = np = 16 \quad -① $$

$$\text{variance} = npq = 8 \quad -②$$

$$\frac{①}{②} \Rightarrow \frac{np}{npq} = \frac{16}{8}$$

$$\Rightarrow \frac{1}{q} = 2 \Rightarrow \boxed{q = \frac{1}{2}}$$

$$\therefore p = 1-q = 1-\frac{1}{2} = \frac{1}{2} \Rightarrow \boxed{p = \frac{1}{2}}$$

from ①, $np = 16$

$$n \cdot \frac{1}{2} = 16$$

$$\Rightarrow \boxed{n = 32}$$

$\therefore$ PMF of Binomial distribution is,

$$P(X=x) = nC_x \, p^x \, q^{n-x}, \quad x = 0, 1, 2 \cdots n$$

$$= 32C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{32-x}, \quad x = 0, 1, 2 \cdots 32$$

Now $P(X=0) = 32C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{32-0}$

$$= \left(\frac{1}{2}\right)^{32}$$

Also $P(X=1) = 32C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{32-1}$

$$= 32 \cdot \left(\frac{1}{2}\right)^{32}$$

$$= \left(\frac{1}{2}\right)^{27}$$

Q) An equipment consists of 5 components each of which may fail independently with probability 0.15. If the equipment is able to function properly when atleast 3 of the components are operational, what is the probability that it functions properly?

Soln:-

n = no. of components = 5

q = Probability that components may fail = 0.15

$\therefore p = 1-q = 1-0.15 = 0.85$, $P(x) = n_{C_x} \, p^x q^{n-x}$

ie $n = 5$, $p = 0.85$, $q = 0.15$

$\therefore$ Prob. that components functions properly

$= P\{x \geq 3\} = P(x=3) + P(x=4) + P(x=5)$

$= 5_{C_3}(\cdot 85)^3(\cdot 15)^{5-3} + 5_{C_4}(\cdot 85)^4(\cdot 15)^{5-4} + 5_{C_5}(\cdot 85)^5(\cdot 15)^{5-5}$

$=$

$=$

Q) If the probability is 0.05 that a certain wide-flange column will fail under a given axial load, what are the probabilities that among 16 such columns

a) atmost three will fail?

b) atleast five will fail?

Soln:

Given $n = 16$, $p = 0.05$, $q = 1 - p = 1 - 0.05 = 0.95$

$$\therefore P(x) = nC_x \, p^x q^{n-x}, \quad x = 0, 1, 2 \ldots n$$

$$= 16C_x \, (.05)^x (.95)^{16-x}, \quad x = 0, 1, 2 \ldots 16$$

a) $P\{\text{atmost three will fail}\}$

$$= P\{X \le 3\}$$

$$= P\{X = 0\} + P\{X = 1\} + P\{X = 2\} + P\{X = 3\}$$

$$= 16C_0 (.05)^0 (.95)^{16-0} + 16C_1 (.05)^1 (.95)^{16-1} + 16C_2 (.05)^2 (.95)^{16-2} + \\ + 16C_3 (.05)^3 (.95)^{16-3}$$

$$= (0.95)^{16} + 16 \times (.05) \times (.95)^{15} + 120 \times (.05)^2 (.95)^{14} + 560 \times (.05)^3 \times (.95)^{13}$$

$$= \underline{\underline{0.9930}}$$

b) $P\{\text{atleast five will fail}\}$

$$= P\{X \geq 5\}$$

$$= 1 - P\{X < 4\}$$

$$= 1 - \left[P(X=0) + P(X=1) + P(X=2) + P(X=3)\right]$$

$$= 1 - \left[16C_0 (.05)^0 (.95)^{16-0} + 16C_1 (.05)^1 (.95)^{16-1} + 16C_2 (.05)^2 (.95)^{16-2} + 16C_3 (.05)^3 (.95)^{16-3}\right]$$

$$= 1 - \left[(.95)^{16} + 16 \times .05 \times (.95)^{15} + 120 \times (.05)^2 \cdot (.95)^{14} + 560 \times (.05)^3 (.95)^{13}\right]$$

$$= 1 - [0.9991]$$

$$= \underline{0.0009}$$

Q) Eight fair coins are tossed 256 times. In how many tosses do you expect atleast one head. ?

Soln:

$n = 8$, $N = 256$, $x \rightarrow$ no. of heads.

In coin tossing experiment, $p = \frac{1}{2}$, $q = \frac{1}{2}$.

$$p(X=x) = nc_x \, p^x \, q^{n-x} , \quad x = 0,1,2 \cdots n$$

$$= 8c_x \left(\tfrac{1}{2}\right)^x \left(\tfrac{1}{2}\right)^{8-x} , \quad x = 0,1,2 \cdots 8.$$

$$P\{\text{atleast one head}\} = P\{X \geq 1\}$$

$$= 1 - P(X<1)$$

$$= 1 - [P(X=0)]$$

$$= 1 - \left[8c_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^{8-0}\right]$$

$$\therefore P(X \geq 1) = 1 - \left(\tfrac{1}{2}\right)^8 = \frac{255}{256}$$

The number of tosses in which atleast one head.

$$= N \times P(X \geq 1)$$

$$= 256 \times \frac{255}{256}$$

$$= 255$$

POISSON DISTRIBUTION

A discrete random variable x is said to be a poisson variable with parameter $\lambda > 0$, if

$$P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!}, \quad x = 0, 1, 2 \ldots$$

ie X follows a poisson distribution with parameter λ and write $X \sim$ poisson (λ).

ie $\quad X \sim P(\lambda)$

Mean and variance :-

$$E(x) = np = \sum x \cdot p(x) = \sum_{x=0} x \cdot \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$= e^{-\lambda} \lambda \sum_{x=0} \frac{\lambda^{x-1}}{(x-1)!} \qquad x! = x(x-1)!$$

$$= \lambda e^{-\lambda} \sum_{x=1} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} \left[\frac{\lambda^{1-1}}{(1-1)!} + \frac{\lambda^{2-1}}{(2-1)!} + \frac{\lambda^{3-1}}{(3-1)!} + \cdots \right]$$

$$= \lambda e^{-\lambda} \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots \right]$$

$$= \lambda e^{-\lambda} \left[e^{\lambda} \right] = \lambda e^{-\lambda+\lambda} = \lambda e^{0} = \lambda$$

$$\Rightarrow \boxed{E(x) = \lambda.} \qquad \because e^{\lambda} = 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots$$

Now variance, $V(x) = E(x^2) - [E(x)]^2$

$$E(x^2) = \sum x^2 \cdot p(x)$$

$$= \sum \left[x(x-1) + x \right] \cdot p(x)$$

$$= \sum_{x=0} x(x-1)\cdot p(x) + \sum_{x=0} x\, p(x)$$

$$= \sum_{x=0} x(x-1)\cdot \frac{e^{-\lambda}\lambda^x}{x!} + \lambda \qquad \because E(x) = \sum x\, p(x) = \lambda.$$

$$= e^{-\lambda} \sum_{x=2} \frac{x(x-1)\lambda^2\cdot\lambda^{x-2}}{x(x-1)(x-2)!} + \lambda$$

$$= \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$= \lambda^2 e^{-\lambda}\left[\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots\right] + \lambda$$

$$= \lambda^2 e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right] + \lambda$$

$$= \lambda^2 e^{-\lambda}\left[e^{\lambda}\right] + \lambda \qquad \because 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots = e^{\lambda}$$

$$E(x^2) = \lambda^2 e^0 + \lambda = \lambda^2 + \lambda$$

$$\Rightarrow \boxed{E(x^2) = \lambda^2 + \lambda}$$

$\therefore$ Variance, $V(x) = E(x^2) - [E(x)]^2$

$$= \lambda^2 + \lambda - (\lambda)^2$$

$$= \lambda$$

$$\Rightarrow \boxed{V(x) = \lambda}$$

For a poisson distribution, mean = Variance = λ

Problems:-

Q) Accidents occur at ~~any~~ an intersection at a poisson rate of 2 per day. what is the probability that in January there are atleast ~~the~~ 3 days without any accidents ? what is the probability that there would be no accidents on a given day ?

Soln:

$X \sim P(\lambda)$

$\lambda =$ poisson rate of accidents occur $= 2$

$$\therefore P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-2}2^{x}}{x!}, \quad x = 0, 1, 2 \ldots$$

i) P{ that in January there are atleast 3 days without any accidents }

$$= P\{X \geq 2\} = 1 - P\{X < 2\}$$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[\frac{e^{-2}2^{0}}{0!} + \frac{e^{-2}2^{1}}{1!}\right]$$

$$= 1 - e^{-2}[1+2] = 1 - 3e^{-2}$$

$$\therefore P(X \geq 2) = 1 - 3e^{-2}$$

ii) P{ that there would be no accidents }

$$= P\{X = 0\}$$

$$= \frac{e^{-2}2^{0}}{0!}$$

$$= e^{-2}$$

Q) ~~Ans~~ The average number of phone calls per minute coming in to a switch board between 2 and 4 PM is 2.5. Determine the probability that during one particular minute there will be

i) 0 ii) 4 or fewer iii) more than 6 telephone calls.

Soln:

$X \sim P(\lambda)$

λ = average number of phone calls = 2.5

$\therefore P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-2.5}(2.5)^{x}}{x!}, \quad x = 0, 1, 2 \cdots$

i) P{ there will be '0' telephone calls }

$= P\{X=0\} = \frac{e^{-2.5}(2.5)^{0}}{0!} = e^{-2.5} =$

ii) P{ 4 or fewer telephone calls }

$= P\{X \leq 4\} = P\{X=0\} + P\{X=1\} + P\{X=2\} + P\{X=3\} + P\{X=4\}$

$= \frac{e^{-2.5}(2.5)^{0}}{0!} + \frac{e^{-2.5}(2.5)^{1}}{1!} + \frac{e^{-2.5}(2.5)^{2}}{2!} + \frac{e^{-2.5}(2.5)^{3}}{3!} + \frac{e^{-2.5}(2.5)^{4}}{4!}$

$= e^{-2.5}\left[1 + 2.5 + \frac{(2.5)^2}{2!} + \frac{(2.5)^3}{3!} + \frac{(2.5)^4}{4!}\right]$

$= e^{-2.5}\left[\qquad\qquad\right] =$

iii) $P\{\text{more than 6 calls}\} = P\{X \geq 6\} = 1 - P\{X < 6\}$

$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=4) + P(X=5)]$

$= 1 - \left[e^{-2.5} + e^{-2.5}(2.5) + \frac{e^{-2.5}(2.5)^2}{2!} + \frac{e^{-2.5}(2.5)^3}{3!} + \frac{e^{-2.5}(2.5)^4}{4!} + \frac{e^{-2.5}(2.5)^5}{5!}\right]$

$=$

H.W
Q) A Geiger counter counts the particle emitted by a radio active material. If the number of particles emitted by per second by a particular radio active material has poisson distribution with a mean of 10 particles, determine the following.

a) the probability that 3 particles are emitted in one second.

b) the probability that more than one particle are emitted in one second.

Soln:-

$$X \sim P(\lambda) \Rightarrow P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2 \ldots$$

$\lambda =$ mean $= 10$

a) $P\{$ 3 particles are emitted in one second $\}$

$$= P\{X=3\} = \frac{e^{-10} (10)^3}{3!} =$$

b) $P\{$more than one particle emitted in one second$\}$

$$= P\{X \geq 1\} = 1 - P\{X < 1\} = 1 - P\{X=0\}$$

$$= 1 - \left[\frac{e^{-10} (10)^0}{0!}\right]$$

$$= \underline{\underline{1 - e^{-10}}}$$

Q) If x is a poisson variate such that $P(x=2) = P(x=3)$, Find $P(x=4)$?

Soln:-

$$x \sim P(\lambda) \Rightarrow P(x=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

Now $P(x=2) = P(x=3)$

$$\Rightarrow \frac{e^{-\lambda}\lambda^{2}}{2!} = \frac{e^{-\lambda}\lambda^{3}}{3!}$$

$$\Rightarrow \frac{\lambda^{2}}{2!} = \frac{\lambda^{3}}{3!}$$

$$\Rightarrow \frac{1}{2!} = \frac{\lambda}{3!} \Rightarrow \lambda = \frac{3!}{2} = \frac{6}{2} = 3$$

$$\Rightarrow \boxed{\lambda = 3}$$

$$\therefore P(x=4) = \frac{e^{-3}(3)^{4}}{4!} = \underline{\underline{0.1680}}$$

Q) If x is a poisson variate such that $P(x=2) = 9\,P(x=4) + 90\,P(x=6)$, Find the mean and variance.

Soln:-

x follows poisson distribution,

$$\therefore P(x=2) = 9\,P(x=4) + 90\,P(x=6)$$

$$\frac{e^{-\lambda}\lambda^{2}}{2!} = 9 \cdot \frac{e^{-\lambda}\lambda^{4}}{4!} + 90 \cdot \frac{e^{-\lambda}\lambda^{6}}{6!}$$

$$\Rightarrow \frac{1}{2!} = \frac{9}{4!}\lambda^{2} + \frac{90}{6!}\lambda^{4}$$

$$\Rightarrow \frac{1}{2} = \frac{9\lambda^2}{1\cdot2\cdot3\cdot4} + \frac{90\lambda^4}{1\cdot2\cdot3\cdot4\cdot5\cdot6}$$

$$\Rightarrow \frac{1}{2} = \frac{3\lambda^2}{8} + \frac{\lambda^4}{8}$$

$$\Rightarrow 1 = \frac{3\lambda^2+\lambda^4}{4} \Rightarrow (\lambda^2)^2 + 3(\lambda^2) - 4 = 0$$

$$\Rightarrow \lambda^2 = \frac{-3 \pm \sqrt{9-4\cdot -4}}{2} = +1, -4$$

$$\Rightarrow \lambda = \pm 1$$

$\Rightarrow \lambda = 1$ since λ is always > 0

$\therefore$ Mean = Variance = 1

POISSON APPROXIMATION TO THE BINOMIAL DISTRIBUTION

※ **Theorem:-**

The probability distribution of a binomial random variable $x = B(n,p)$ approaches that of a poisson random variable, poisson (λ), as $n \to \infty$ and $p \to 0$ such that $np = \lambda$, a constant.

Proof:-

Binomial p.M.F, $P\{X = x\} = nc_x\, p^x q^{n-x}$

$$= \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \quad , \quad np = \lambda \Rightarrow p = \frac{\lambda}{n}$$

$$= \cancel{\frac{n(n-1)(n-2)\,(n-x)[n-(x+1)]}{x!\,(n-x)!}} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$\Rightarrow \frac{n(n-1)(n-2)\cdots(n-x+1)(n-x)!}{x!\,(n-x)!}\left(\frac{\lambda}{n}\right)^{x}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$\Rightarrow \frac{n(n-1)\cdots(n-x+1)}{x!}\,\frac{\lambda^{x}}{n^{x}}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$\Rightarrow \frac{n^{x}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(x-1)}{n}\right)\right]}{x!}\,\frac{\lambda^{x}}{n^{x}}\left(1-\frac{\lambda}{n}\right)^{n}\cdot\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$\Rightarrow \frac{\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(x-1)}{n}\right)\right]}{x!}\,\lambda^{x}\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-x}$$

when $n\to\infty$, $\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots 1-\frac{(x-1)}{n} \to 1.$

and $\lim\limits_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n-x} = \lim\limits_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}\cdot\left(1-\frac{\lambda}{n}\right)^{-x} \to e^{-\lambda}$

$\therefore$ as $n\to\infty$,

$$P\{x\} = \frac{[1.]\cdot\lambda^{x}\cdot e^{-\lambda}}{x!}$$

$$P\{X=x\} = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x=0,1,2\cdots$$ which is the PMF of poisson distribution.

$\therefore$ B.D tends to P.D when $n\to\infty$ and $np=\lambda$.

problems:-

In a given city, 6% of all drivers get atleast one parking ticket per year. Use poisson distribution determine the probability that among 80 drivers,

i) four will get atleast one parking ticket.

ii) atleast ~~one~~ ~~parking~~ 3 will get one parking ticket

iii) anywhere from 3 to 6, will get atleast one ticket.

Soln:

$n = 80, \quad p = 6\% = 0.06 = \frac{6}{100}$

Using binomial approximation to poisson, $np = \lambda$

ie $\lambda = n \cdot p = 80 \times \frac{6}{100} = 4.8$

ie $\boxed{\lambda = 4.8}$ $\quad \therefore P(x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-4.8}(4.8)^x}{x!}$

i) $P\{\text{four will get atleast one ticket}\}$

$= P\{x = 4\} = P\{x=4\} = \frac{e^{-4.8}(4.8)^4}{4!} = 0.182$

ii) $P\{\text{atleast 3 will get parking ticket}\}$

$= P(x \geq 3) = 1 - P(x<3) = 1 - [P(x=0) + P(x=1) + P(x=2)]$

$= 1 - \left[\frac{e^{-4.8}(4.8)^0}{0!} + \frac{e^{-4.8}(4.8)^1}{1!} + \frac{e^{-4.8}(4.8)^2}{2!}\right]$

$= 1 - 0.143 = 0.857$

iii) $P\{\text{anywhere from 3 to 6}\} = P\{3 \leq x \leq 6\}$

$= P\{x=3\} + P\{x=4\} + P\{x=5\} + P\{x=6\}$

$= \frac{e^{-4.8}(4.8)^3}{3!} + \frac{e^{-4.8}(4.8)^4}{4!} + \frac{e^{-4.8}(4.8)^5}{5!} + \frac{e^{-4.8}(4.8)^6}{6!} = 0.648$

Q) It is known that 2% of the bolts produced by a company are defective. The bolts are supplied in boxes of 200 bolts. what is the probability that a randomly chosen box contain no more than 5 defective bolts. ?

Soln:

$p =$ prob. that bolts are defective $= \frac{2}{100} = 0.02.$

$n =$ number of bolts $= 200$

Using binomial approximation to poisson,

$np = \lambda \Rightarrow 200 \cdot \frac{2}{100} = \lambda \Rightarrow \boxed{\lambda = 4}$

$\therefore P(x) = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-4} 4^x}{x!}, \quad x = 0, 1, 2 \ldots$

Now $P\{$ box contain no more than 5 defective bolt $\}$

$= P\{X \leq 5\}$ ~~$P\{X=5\} + P\{X=6$~~

$= P\{X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$$= \frac{e^{-4} 4^0}{0!} + \frac{e^{-4} 4^1}{1!} + \frac{e^{-4} 4^2}{2!} + \frac{e^{-4} 4^3}{3!} + \frac{e^{-4} 4^4}{4!} + \frac{e^{-4} 4^5}{5!}$$

$$= e^{-4}\left[1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!}\right]$$

$P(X \leq 5) = \underline{\underline{0.785}}$

DISCRETE BIVARIATE DISTRIBUTIONS

Suppose X and Y are discrete random variables. Then the ordered pair (X,Y) is called discrete ~~random vari~~ bivariate random variables.

$$P\{x, y\} = \begin{cases} P(X = x, Y = y) \end{cases}$$

is called joint probability mass function (joint PMF) of (X,Y).

i) $\sum P(x,y) = \sum P(X=x, Y=y) = 1.$

ii) $P(x,y) \geq 0$

MARGINAL DISTRIBUTIONS

$$\boxed{\begin{aligned} P(X=x) &= \sum_{y} P(x,y) \text{ (marginal PMF of X)} \\ P(Y=y) &= \sum_{x} P(x,y) \text{ (marginal PMF of Y)} \end{aligned}}$$

INDEPENDANCE

R.V X and Y are said to be independent if,

$$\boxed{P(x,y) = P_X(x) \times P_Y(y)}$$

problems

1) a The joint probability mass function of random variable X and Y is given by,

$$P(x,y) = k(x+2y) \quad , x = 0,1,2 \ ; \ y = 0,1,2,3$$

Find the following.

i) k ii) $P(X \leq 1)$ iii) $P(Y \leq 2)$

iv) $P(X \leq 1, Y \leq 2)$ v) $P(X \leq 1 / Y \leq 2)$ vi) $P(X+Y \leq 3)$

vii) the marginal distribution of X and Y.
Are X and Y are independent.

Soln:-

The joint probability distribution can be obtained as

X \ Y	Y=0	Y=1	Y=2	Y=3
X=0	0	2k	4k	6k
X=1	k	3k	5k	7k
X=2	2k	4k	6k	8k

i) The value of k is obtained from,

$$\sum_x \sum_y P(x,y) = 1$$

$$\Rightarrow 0+2k+4k+6k+k+3k+5k+7k+2k+4k+6k+8k = 1$$

$$\Rightarrow 48k = 1 \Rightarrow \boxed{k = \frac{1}{48}}$$

∴ Joint PMF is,

X \ Y	Y=0	Y=1	Y=2	Y=3
X=0	0	$\frac{2}{48}$	$\frac{4}{48}$	$\frac{6}{48}$
X=1	$\frac{1}{48}$	$\frac{3}{48}$	$\frac{5}{48}$	$\frac{7}{48}$
X=2	$\frac{2}{48}$	$\frac{4}{48}$	$\frac{6}{48}$	$\frac{8}{48}$

ii) To evaluate $P(X \le 1)$, we sum up the probabilities corresponding to $X = 0, 1$.

$$\therefore P(X \le 1) = P(0,0) + P(0,1) + P(0,2) + P(0,3) + P(1,0) + P(1,1) + P(1,2) + P(1,3)$$

$$= 0 + \frac{2}{48} + \frac{4}{48} + \frac{6}{48} + \frac{1}{48} + \frac{3}{48} + \frac{5}{48} + \frac{7}{48}$$

$$= \frac{28}{48} = \frac{7}{12}$$

iii) $P(Y \le 2)$ is the sum of probabilities corresponding to the values of $Y = 0, 1, 2$.

$$P(Y \le 2) = P(0,0) + P(1,0) + P(2,0) + P(0,1) + P(1,1) + P(2,1) + P(0,2) + P(1,2) + P(2,2)$$

$$= 0 + \frac{1}{48} + \frac{2}{48} + \frac{2}{48} + \frac{3}{48} + \frac{4}{48} + \frac{4}{48} + \frac{5}{48} + \frac{6}{48}$$

$$= \frac{27}{48} = \frac{9}{16}$$

iv) $P(X \le 1, Y \le 2) = P(X = 0, 1; Y = 0, 1, 2)$

$$= P(0,0) + P(0,1) + P(0,2) + P(1,0) + P(1,1) + P(1,2)$$

$$= 0 + \frac{2}{48} + \frac{4}{48} + \frac{1}{48} + \frac{3}{48} + \frac{5}{48}$$

$$= \frac{15}{48} = \frac{5}{16}$$

v) $P(X \le 1 / Y \le 2) = \dfrac{P(X \le 1, Y \le 2)}{P(Y \le 2)}$

$$= \frac{\frac{5}{16}}{\frac{9}{16}} = \frac{5}{9}$$

vi) $P(X+Y \le 3)$

$$= P(0,0) + P(0,1) + P(0,2) + P(0,3) + P(1,0) + P(1,1) + P(1,2) + P(2,0) + P(2,1)$$

$$= 0 + \frac{2}{48} + \frac{4}{48} + \frac{6}{48} + \frac{1}{48} + \frac{3}{48} + \frac{5}{48} + \frac{2}{48} + \frac{4}{48}$$

$$= \frac{27}{48} = \frac{9}{16}$$

vii) The marginal probabilities of X are obtained from right margin of table below. ie $P_X(x)$.

The marginal probabilities of Y are obtained from bottom margin.

X \ Y	Y=0	Y=1	Y=2	Y=3	$P_X(x)$
X=0	0	$\frac{2}{48}$	$\frac{4}{48}$	$\frac{6}{48}$	$12/48$
X=1	$\frac{1}{48}$	$\frac{3}{48}$	$\frac{5}{48}$	$\frac{7}{48}$	$16/48$
X=2	$\frac{2}{48}$	$\frac{4}{48}$	$\frac{6}{48}$	$\frac{8}{48}$	$20/48$
$P_Y(y)$	$\frac{3}{48}$	$\frac{9}{48}$	$\frac{15}{48}$	$\frac{21}{48}$	

Marginal distribution of X;

X :	0	1	2
$P_X(x) = P(X=x)$:	$\frac{12}{48}$	$\frac{16}{48}$	$\frac{20}{48}$

Marginal distribution of Y;

Y :	0	1	2	3
$P_Y(y) = P(Y=y)$:	$\frac{3}{48}$	$\frac{9}{48}$	$\frac{15}{48}$	$\frac{21}{48}$

viii) For testing Independance, we check

~~$P_X(x) \times P_Y(y) = P_X(0) \times P_Y(0) = 0 \times 0 = 0 = P(0,0)$~~

~~But~~ $\boxed{P_X(x) \times P_Y(y) = P(x,y)}$

for ex: $P_X(0) \times P_Y(0) = P(0,0) = 0$ satisfied.

But $P_X(1) \times P_Y(1) = \frac{16}{48} \times \frac{9}{48} = \frac{144}{48}$

also $P(1,1) = \frac{3}{48}$

~~clearly $P_X(0) \neq 0$ $P_X(0) \times P_Y(0) = \frac{144}{48} \neq$~~

clearly $P_X(1) \times P_Y(1) = \frac{144}{48} \neq P(1,1)$.

$\therefore$ X and Y are not independent.

H.W 1) The joint PMF of two random variables X and Y is given by $P(x,y) = k(2x+y)$, $x = 1,2$; $y = 1,2$ where k is constant.

i) Find k

ii) Find the marginal density functions of X & Y.

iii) Are X and Y independent. ?

2) The joint PMF of two discrete random variables X and Y is given by,

$$P(x,y) = kxy, \quad x = 1,2,3 \quad ; \quad y = 1,2,3$$

Find i) k ii) $P(1 \le X \le 2, Y \le 2)$

iii) the marginal distributions

iv) Are X and Y independent ?

EXPECTATION OF BIVARIATE DISTRIBUTION

$$E(x) = \sum_{x} x\, P_X(x)$$

$$E(y) = \sum_{y} y\, P_Y(y)$$

$$E(xy) = \sum_{x}\sum_{y} xy\, P(x,y)$$

Problems:-

1) The joint probability mass function of random variables X and Y is given by,

$$P(x,y) = \begin{cases} \dfrac{x(x+y)}{70}, & x=1,2,3\ ;\ y=3,4 \end{cases}$$

Find i) $E(x)$ ii) $E(y)$ iii) $E(xy)$

Soln:

Given $P(x,y) = \dfrac{x(x+y)}{70}$, $x=1,2,3\ ;\ y=3,4$

X \ Y	Y=3	Y=4	$P_X(x)$
X=1	$\frac{4}{70}$	$\frac{5}{70}$	$9/70$
X=2	$\frac{10}{70}$	$\frac{12}{70}$	$22/70$
X=3	$\frac{18}{70}$	$\frac{21}{70}$	$39/70$
$P_Y(y)$	$\frac{32}{70}$	$38/70$	1

Marginal PMF of X is,

X :	1	2	3
P(X)	$9/70$	$22/70$	$39/70$

$$\therefore E(X) = \sum_{x} x P_X(x)$$

$$= 1 \cdot \frac{9}{70} + 2 \cdot \frac{22}{70} + 3 \cdot \frac{39}{70}$$

$$= \frac{170}{70} = \frac{17}{7}$$

Marginal PMF of Y, $P_Y(y)$ is,

Y :	3	4
$P_Y(y)$:	$\frac{32}{70}$	$\frac{38}{70}$

$$E(Y) = \sum_{y} y \cdot P_Y(y)$$

$$= 3 \cdot \frac{32}{70} + 4 \cdot \frac{38}{70}$$

$$= \frac{248}{70}$$

$$E(XY) = \sum_{x} \sum_{y} x \cdot y \cdot P(x, y)$$

$$= 1 \cdot 3 \cdot \frac{4}{70} + 1 \cdot 4 \cdot \frac{5}{70} + 2 \cdot 3 \cdot \frac{10}{70} + 2 \cdot 4 \cdot \frac{12}{70} + 3 \cdot 3 \cdot \frac{18}{70} + 3 \cdot 4 \cdot \frac{21}{70}$$

$$= \frac{602}{70}$$

H.W The joint PMF of X and Y is are given in following table.

X \ Y	$Y=0$	$Y=1$	$Y=2$
$X=-1$	$1/15$	$2/15$	$1/15$
$X=0$	$3/15$	$2/15$	$1/15$
$X=1$	$2/15$	$1/15$	$2/15$

i) Compute $E(X)$ and $E(Y)$

ii) $E(XY)$.

CONTINUOUS RANDOM VARIABLE

If the sample space consists of infinite no. of pts or it takes all the values in an interval, then it is said to be continuous sample space & similarly if a random variable is ~~as~~ takes all the values in an interval, then it is said to be a continuous random variable.

Probability mass fn. / Probability density fn.

The probability fn. $f(x)$ for a continuous random variable x is said to be a pmf if

(i) $f(x) \geq 0 \quad \forall x$ &

(ii) $\int_{-\infty}^{+\infty} f(x)\,dx = 1$.

Distribution fn / Cumulative fn

The distribution fn. / cumulative fn, $F(x)$ for a pmf $f(x)$ is defined as,

$$F(x) = P[X \leq x]$$

$$= \int_{-\infty}^{x} f(x)\,dx$$

Mean & Variance

Let X be a continuous random variable take all the values in the interval (a,b) with pmf $f(x)$, then mean,

$$E(X) = \int_a^b x f(x)\,dx.$$

Variance, $V(X) = E(X^2) - [E(X)]^2$

Q. A CRV X has a pmf $f(x) = k\cancel{x}(1+x)$ $\quad 2 \le x \le 5$

Find $P(X \le 4)$.

Ans: $\because f(x)$ is a pdf, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Here $\int_{2}^{5} k(1+x)\,dx = 1$

$$k.\left[x + \frac{x^2}{2}\right]_2^5 = 1$$

$$k\left[5 + \frac{25}{2} - 2 - 2\right] = 1$$

$$k = \frac{2}{27}$$

$$\therefore f(x) = \frac{2}{27}(1+x).$$

$$P(X \le 4) = \int_{2}^{4} \frac{2}{27}(1+x)\,dx$$

$$= \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^4$$

$$= \frac{16}{27} = 0.592$$

Q. Find c for which $f(x) = c\,x\,e^{-x}$ $\quad 0 \le x \le \infty$

$= 0$ $\quad$ otherwise

is a pdf.

Ans: $\int_{-\infty}^{\infty} f(x)dx = 1$ $\qquad \int uv = u'v_1 \mp u''v_2 + - - - -$

Here, $\int_0^{\infty} c\, x\, e^{-x} dx = 1$

$$c\left[x.\frac{e^{-x}}{-1} - 1\frac{e^{-x}}{(-1)(-1)}\right]_0^{\infty} = 1$$

$c(0-1) = 1 \qquad \Rightarrow \underline{\underline{c = 1}}$.

Q. The pdf of x is given by $f(x) = y_0 e^{-|x|} \quad -\infty < x < \infty$.
Find the mean & variance.

Ans: $\int_{-\infty}^{\infty} y_0 e^{-|x|} dx = 1 \qquad \Rightarrow 2\int_0^{\infty} y_0 e^{-x} dx = 1$

$\cancel{y_0\left[\frac{e^{-x}}{-1}\right]_{-\infty}^{\infty} = 1}$ $\qquad 2 y_0\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$

$\cancel{-y_0(0-0) = 1}$

$-2y_0[0-1] = 1$

$2y_0 = 1$

$\underline{\underline{y_0 = 1/2}}$.

$$\text{Mean} = \int_{-\infty}^{\infty} \frac{x}{2} e^{-|x|} dx = 0 \quad \because x\, e^{-|x|} \text{ is an odd fn.}$$

$V(x) = E(x^2) - (E(x))^2$

$$E(x^2) = \int_{-\infty}^{\infty} \frac{x^2}{2} e^{-|x|} dx$$

$$= 2\int_0^{\infty} \frac{x^2}{2} e^{-|x|} dx$$

$$= \int_0^{\infty} x^2 e^{-x} dx = \left[x^2.\frac{e^{-x}}{-1} - 2x.\frac{e^{-x}}{(-1)(-1)} + 2.\frac{e^{-x}}{(-1)}\right]_0^{\infty}$$

$$= \left[x^2 e^{-x} - x e^{-x} - 2e^{-x}\right]_0^{\infty} = 2$$

Q. The pdf of a CRV x is given to be

$$f(x) = kx(1-x) \quad 0 \le x \le 1$$
$$= 0 \quad \text{elsewhere}$$

i) Find k

ii) $P(X \le 1/2)$

iii) Find dist fn.

Ans: i) $\int_0^1 kx(1-x)\,dx = 1$

$$k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$k\left[\frac{1}{2} - \frac{1}{3}\right] = 1$$

$$k = 6$$

ii) $P(X \le 1/2) = \int_0^{1/2} 6x(1-x)\,dx$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{1/2}$$

$$= 6\left[\frac{1}{8} - \frac{1}{24}\right]$$

$$= \frac{6 \times 16}{24 \times 8} = 1/2$$

iii) $F(x) = P(X \le x)$

$$= \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_0^x f(x)\,dx$$

$$= 0 + 6\int_0^x (x - x^2)\,dx$$

KTUNOTES.IN

$$= 0+6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^x$$

$$= 6\left[\frac{x^2}{2}-\frac{x^3}{3}\right] \quad 0<x<1$$

$$= 3x^2-2x^3$$

$$F(x) = 0 \qquad x<0$$
$$= 3x^2-2x^3 \quad 0<x<1$$
$$= 1 \qquad x\geq 1$$

03.2017

A continuous random variable satisfying the pdf

$$f(x) = \frac{1}{b-a} \;;\; a\leq x\leq b$$
$$= 0 \qquad ; \text{ otherwise}$$

is said to be a uniform dist. and is denoted as $u(a,b)$.

Uniform dist is also known as the rectangular dist.

Mean & variance

Mean = E(X)

$$= \int_a^b x\, f(x)\,dx$$

$$= \int_a^b x \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$E(X) = \frac{1}{2(b-a)}(b^2-a^2) = \frac{b+a}{2}$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$E(x^2) = \int_a^b x^2 f(x)\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{b-a}\,\frac{b^3-a^3}{3}$$

$$= \frac{1}{3\cancel{(b-a)}}\cancel{(b-a)}(b^2+ab+a^2)$$

$$= \frac{b^2+ab+a^2}{3}$$

$$[E(x)]^2 = \frac{(b+a)^2}{4}$$

$$\therefore V(x) = \frac{b^2+ab+a^2}{3} - \frac{b^2+2ab+a^2}{4}$$

$$V(x) = \frac{b^2-2ab+a^2}{12} = \frac{(a-b)^2}{12}$$

Q. A random variable x has a uniform dist. in $(-3,3)$. Find i) $P(x<2)$ (ii) $P(|x|<2)$ (iii) $P(|x-2|<2)$. And k for which $P(X>k) = 1/3$.

Ans: i) $f(x) = \frac{1}{3-(-3)} = \frac{1}{6}$

$$P(X<2) = \int_{-3}^{2} \frac{1}{6}\,dx = 5/6$$

(ii) $P(|x|<2)$. $\quad -2 \le x \le 2$.

$$= \int_{-2}^{2} \frac{1}{6}\,dx$$

$$= \underline{\underline{4/6}}.$$

(iii) $P(|x-2|<2) = P(-2 \le x-2 \le 2)$

$$= P(0 \le x \le 4).$$

$$= \int_0^3 \frac{1}{6}\,dx + \int_3^4 0\,dx$$

$$= \frac{1}{6} \times 3$$

$$= \underline{\underline{1/2}}.$$

(iv) $P(x>k) = 1/3$

$$\int_k^3 f(x)\,dx \cancel{= 1/3} = 1/3$$

$$\frac{1}{3} = \int_k^3 1/6\,dx$$

$$\frac{1}{\cancel{3}} = \frac{1}{\cancel{6}_{\,2}}(3-k).$$

$3-k = 2$

$\underline{\underline{k=1}}$.

Q. If X is a uniformly dist random variable with mean 1, variance $4/3$. Find $P(X<0)$.

Ans: $E(X) = \frac{b+a}{2} = 1.$

$b+a = 2$ —— ①.

$V(X) = \frac{(b-a)^2}{12} = \frac{4}{3}$

$(b-a)^2 = 16$

$b-a = \pm 4$

$b-a = 4$ —— ②

① + ② ⇒

$2b = 6.$

$\underline{\underline{b=3}}.$

$\underline{\underline{a=-1}}.$

$-1 \leq x \leq 3.$

$f(x) = \frac{1}{b-a} = \frac{1}{3+1} = \frac{1}{4}$ //.

$P(X<0) = \int_{-1}^{0} \frac{1}{4} dx.$

$= \frac{1}{4}(0+1) = \underline{\underline{\frac{1}{4}}}.$

Q. A bus arrives every 15 mins at a busstop. Assuming that the waiting time x for bus is uniformly distributed find the probability that a person has to wait for the bus i) more than 10 min
ii) b/w 5 & 10 mins

Ans: Interval (0,15)

$$f(x) = \frac{1}{b-a} = \frac{1}{15}$$

i) $\int_{10}^{15} \frac{1}{15} dx = \frac{1}{3}$

ii) $P(5 \le x \le 10) = \int_{5}^{10} 1/15 \, dx$

$= 1/3$.

Q. A rand vari Y is defined as $\cos\pi x$ where x ~~takes~~ has a uniform dist over $(-1/2, 1/2)$. Find the mean & variance of Y

Ans: $f(x) = \frac{1}{b-a} = \frac{1}{1/2 + 1/2} = 1$.

$y = \cos\pi x$.

$\cancel{E(y) = \int_a^b f \quad f f(x) dx}$.

$$E(Y) = \int_{-1/2}^{1/2} \cos\pi x . 1 \, dx$$

$$= \left[\frac{\sin\pi x}{\pi}\right]_{-1/2}^{1/2} = \frac{2}{\pi}$$

$$E(y^2) = \int_{-1/2}^{1/2} \cos^2 \pi x \cdot 1 \, dy$$

$$= \int_{-1/2}^{1/2} \frac{1+\cos 2\pi x}{2} \, dy$$

$$= \frac{1}{2}\left[y + \frac{\sin 2\pi x}{2\pi} \right]_{-1/2}^{1/2}$$

$$= \frac{1}{2}[1+0]$$

$$= 1/2$$

$$V(Y) = E(Y^2) - [E(Y)]^2$$

$$= 1/4 - \frac{4}{\pi^2}$$

06/03/2017

Normal distribution

A continuous random variable satisfying

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{where} \quad -\infty \le x \le \infty$$

is said to be a normal dist where,

$\mu \rightarrow$ Arithmetic mean

$\sigma \rightarrow$ std deviation.

and is denoted as $N(\mu, \sigma)$.

Standard Normal dist

A normal dist with $\mu=0$ & $\sigma=1$ is said to be a std normal dist and is denoted as $N(0,1)$.

An $N(\mu,\sigma)$ can be converted to an $N(0,1)$ by the substitution $z = \frac{x-\mu}{\sigma}$.

Properties of normal dist

i) The normal curve is bell shaped.

ii) The max area under the normal curve is unity

i.e. $\int_{-\infty}^{\infty} f(x)dx = 1$

iii) The max ordinate is attained at the arithmetic mean

i.e. $f(x) = \frac{1}{\sqrt{2\pi}\sigma}$

iv) The arithmetic mean, median & mode coincides.

v) The normal curve is symmetric about the arithmetic mean. i.e.

$$\int_{-\infty}^{0} f(x)dx = \int_{0}^{\infty} f(x)dx.$$

Q. Find the mean & variance of the normal dist given by $y = \frac{1}{\sqrt{104\pi}} e^{x^2/104}$ $\quad -\infty < x < \infty$

Ans: comparing with pdf $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$,

$\mu = 0$,

$2\sigma^2 = 104$

$\sigma^2 = 52$

$\sigma = \sqrt{52}$ //.

Q. For a normally dist variant with mean 1 & std dev 3 find the probability that $3.43 \le x \le 6.19$

Ans: $\mu = 1$

$sd, \sigma = 3$

$$\int_{3.43}^{6.19} f(x)\,dx = P(3.43 \le x \le 6.19)$$

$$N(1,3) \longrightarrow N(0,1)$$

$$z = \frac{x-\mu}{\sigma}$$

At $x = 3.43$,

$$z = \frac{3.43-1}{3}$$

$$= \frac{2.43}{3}$$

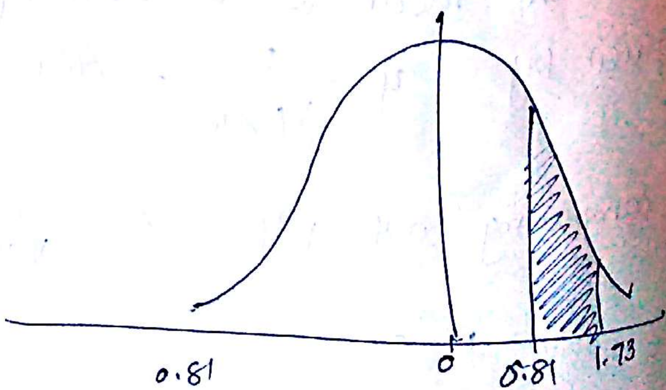
$$= \underline{0.81}$$

At $x = 6.19$

$$z = \frac{6.19-1}{3}$$

$$z = \underline{\underline{1.73}}$$

$$\therefore \int_{3.43}^{6.19} f(x) = \int_{0.81}^{1.73} f(z)\,dz$$



$$\therefore \int_{0.81}^{1.73} f(z)\,dz = \int_{0}^{1.73} f(z)\,dz - \int_{0}^{0.81} f(z)\,dz$$

$= 0.4582 - 0.2910$

$= 0.1672$

8. A savings bank account has avg balance of ₹ 150. and std. deviation of 50. Assuming account balances are normally distributed, find what % account is
i) over 200 ₹
ii) b/w ₹120 & ₹170
iii) less than ₹75.

Ans: $\mu = 150$

$\sigma = 50$

$$N(150,50) \longrightarrow N(0,1)$$

$$z = \frac{x-\mu}{\sigma}$$

i) $x = 200,$

$$z = \frac{200-150}{50}$$

$$z = 1$$

$$\int_1^\infty f(z)\,dz = \int_0^\infty f(z)\,dz - \int_0^1 f(z)\,dz = 0.5 - 0.3413 = 0.1587$$

$= 15.87\%$

ii) $x = 120$

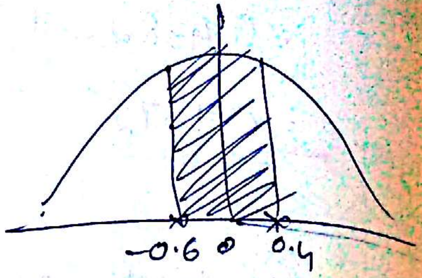
$$z = \frac{\cancel{200-120}\ 120-150}{50} = \frac{\cancel{80}\ -30}{50} = \frac{3}{5} = 0.6$$

$x = 170$

$$z = \frac{\cancel{200-170}\ 170-150}{50} = \frac{\cancel{30}\ 20/50}{50} = \frac{2}{5} = 0.4$$

$$\cancel{\int_{1.6}^{0.6} f(z)\,dz = \int_{0}^{0.6} f(z)\,dz - \int_{0}^{1.6} f(z)\,dz}$$

$$\cancel{= 0.2257} -$$



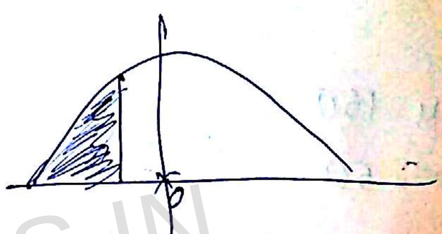
$$\int_{-0.6}^{0.4} f(z)\,dz = \int_{0}^{0.4} f(z)\,dz + \int_{0}^{-0.6} f(z)\,dz.$$

$$= \int_{0}^{0.4} f(z)\,dz + \int_{0}^{0.6} f(z)\,dz = 0.1554 + 0.2257$$

$$= \underline{\underline{0.3811}} \qquad \underline{\underline{38.11\%}}$$

c) $\underline{\underline{x = 75}}$

$$z = \frac{75-150}{50} = -1.5$$



$$\int_{-\infty}^{75} f(x)\,dx = \int_{-\infty}^{0} f(z)\,dz + \int_{0}^{-1.5} f(z)$$

$$= 0.5 \cancel{+} - 0.4332$$

$$= \underline{\underline{\cancel{0.9332}}} = \underline{\underline{0.0668}}$$

07/03/16

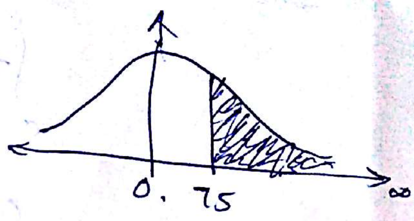
Q. Marks obtained by a no. of students in a certain subject are approx normally distributed with mean 65 and sd 5. If 3 students are selected randomly from this grp, what is the probability that atleast one of them would have scored above 75?

Ans: $\mu = 65$

$\sigma = 5$

$z = \frac{75-65}{5} = 2$

$$P(x>75) = \int_{75}^{\infty} f(x)\,dx = \int_{2}^{\infty} f(z)\,dz$$



$$P = \int_{0}^{\infty} f(z)\,dz - \int_{0}^{2} f(z)\,dz$$

$P = 0.5 - 0.4772$

$P = 0.0228$

$q = 1 - 0.0228 = 0.9772$

$P(x \geq 1) = 1 - P(x<1)$

$= 1 - P(0)$

$P(0) = {}^3C_0\, p^0 q^3$

$= 1.(0.9331)$

$= 0.9331$

$P(x \geq 1) = 1 - P(0)$

$= 1 - 0.9331$

$= 0.0668$

Q. If x is normally distributed with mean 20 & sd 5 find the probabilities of (i) $x \geq 35$, (ii) $x \leq 30$, (iii) $10 \leq x \leq 40$, (iv) $|x-20| > 10$

Ans:

$\mu = 20$

$sd = 5$

i) $P(x \geq 35)$ $\quad z=35 \quad z = \frac{35-20}{5} = 3$

$$P(X>30)=\int_{35}^{\infty} f(x)\,dx = \int_{3}^{\infty} f(z)\,dz$$

$$= \int_{0}^{\infty} f(z)\,dz - \int_{0}^{3} f(z)\,dz$$

$$= 0.5 - 0.4983 = \underline{\underline{0.0013}}$$

ii) $P(X \leq 30) = \int_{-\infty}^{30} f(x)\,dx$ $\qquad z = \frac{30-20}{5} = 2$

$$= \int_{-\infty}^{2} f(z)\,dz$$

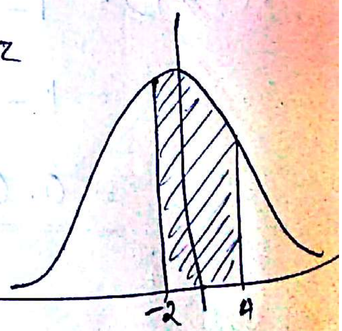
$$= \int_{-\infty}^{0} f(z)\,dz + \int_{0}^{2} f(z)\,dz$$

$$= 0.5 + 0.4772$$

$$= \underline{\underline{0.9772}}$$

iii) $P(10 \leq x \leq 40) = \int_{10}^{40} f(x)\,dx = \int_{-2}^{4} f(z)\,dz$ $\qquad z = \frac{10-20}{5} = -2$ $\qquad z = \frac{40-20}{5} = 4$

$$= \int_{0}^{+2} f(z)\,dz + \int_{0}^{4} f(z)\,dz$$



$$= 0.4772 + 0.4990$$

$$= \underline{\underline{0.9762}}$$

iv) $P(|x-20|>10)$

$$= 1 - P(|x-20| \leq 10)$$

$$= 1 - P[-10 \leq x-20 \leq 10]$$

$$= 1 - P[10 \leq x \leq 30]$$

$$= 1 - \int_{-2}^{2} f(z)\,dz$$

$$\cancel{= 1 - [0.227719]} = 1 - 2\int_{0}^{2} f(z)\,dz$$

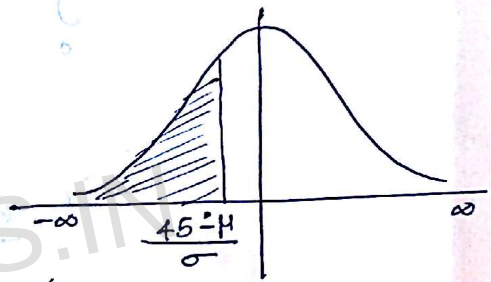
$$\cancel{= 0.7723} \qquad = 1 - 0.9544$$

$$= \underline{0.0456}$$

Q. In a normal dist 31% of items are under 45 & 8% over 64 find mean & sd of the dist.

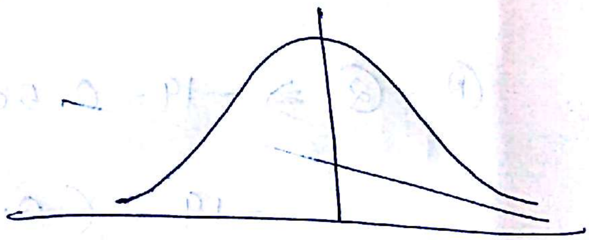
ans: $P(X<45) = 31\% = 0.31$

$$\Rightarrow \int_{-\infty}^{\frac{45-\mu}{\sigma}} f(z)\,dz = 0.\cancel{0}31.$$



$P(X>64) = 8\% = 0.08$

$$\Rightarrow \int_{\frac{64-\mu}{\sigma}}^{\infty} f(z)\,dz = 0.08$$



$$\int_{-\infty}^{\frac{45-\mu}{\sigma}} = \int_{-\infty}^{0} - \int_{0}^{\frac{45-\mu}{\sigma}} = 0.\cancel{0}31$$

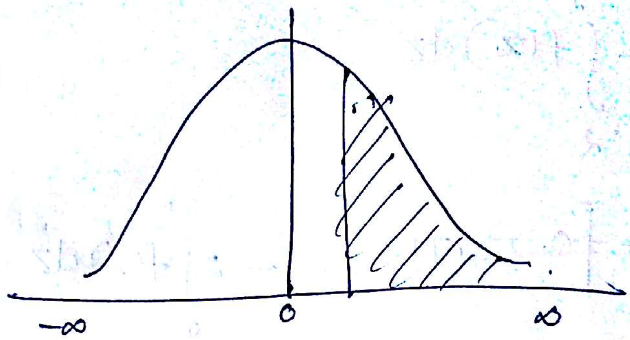
$$\Rightarrow 0.5 - \int_{0}^{\frac{45-\mu}{\sigma}} = 0.\cancel{0}31$$

$$\Rightarrow \int_{0}^{\frac{45-\mu}{\sigma}} = 0.5 - 0.31$$

$$= \underline{0.19}, \qquad \text{i.e. } \frac{45-\mu}{\sigma} = 0.5$$

$$\rightarrow 45 - \mu = 0.5\sigma \quad \text{---} (1)$$

$$\int_{\frac{64-\mu}{\sigma}}^{0} f(z)\,dz = 0.08$$



$$\Rightarrow \int_{0}^{\infty} f(z)\,dz - \int_{0}^{\frac{64-\mu}{\sigma}} = 0.08$$

$$0.5 - \int_{0}^{\frac{64-\mu}{\sigma}} = 0.08$$

$$\int_{0}^{\frac{64-\mu}{\sigma}} = 0.5 - 0.08$$

$$= \cancel{0.58}\ \underline{0.42}$$

i.e. $\dfrac{64-\mu}{\sigma} = 1.41$

$$64-\mu = 1.41\sigma \quad — (2)$$

$$(1) - (2) \Rightarrow -19 = 0.5\sigma - 1.41\sigma$$

$$-19 = (-0.91)\sigma$$

$$\sigma = \frac{19}{0.91}$$

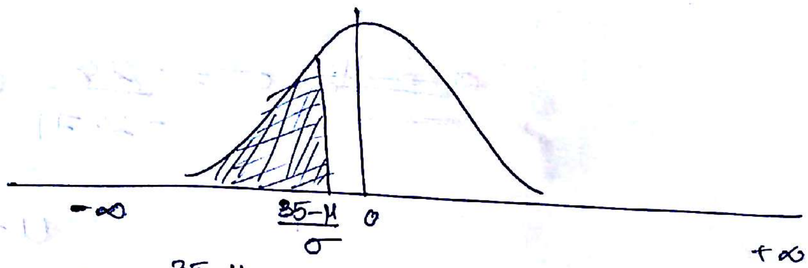
$\sigma = 9.947 \approx 10$

$\mu = 50$

08/03/17

Q. In a normal dist. 7% of the items are under 35 and 89% are under 63. What are the mean & sd of the dist.?

Ans: $P(X<35) = 7\% = 0.07 = \int_{-\infty}^{\frac{35-\mu}{\sigma}} f(z)\,dz$



$$\int_{-\infty}^{\frac{35-\mu}{\sigma}} f(z) = \int_{-\infty}^{0} f(z)\,dz - \int_{\frac{35-\mu}{\sigma}}^{0} f(z)\,dz$$

$$0.07 = 0.5 - \int_{0}^{\frac{35-\mu}{\sigma}} f(z)\,dz$$

$$\Rightarrow \int_{0}^{\frac{35-\mu}{\sigma}} f(z)\,dz = 0.5 - 0.07$$

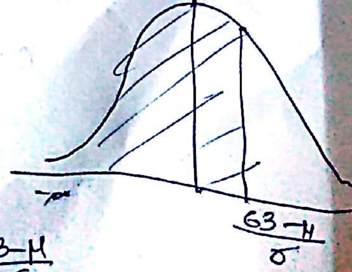
$$= 0.43$$

$$\frac{35-\mu}{\sigma} = -1.48$$

$$35-\mu = -1.48\sigma \quad \text{— (1)}$$

$$P(X<63) = 89\% = 0.89 = \int_{-\infty}^{\frac{63-\mu}{\sigma}} f(z)\,dz$$

$$\int_{-\infty}^{\frac{63-\mu}{\sigma}} f(z)\,dz = \int_{-\infty}^{0} f(z)\,dz + \int_{0}^{\frac{63-\mu}{\sigma}} f(z)\,dz$$



$$0.89 = 0.5 + \int_{0}^{\frac{63-\mu}{\sigma}} f(z)\,dz \Rightarrow \int_{0}^{\frac{63-\mu}{\sigma}} f(z)\,dz = 0.89 - 0.5$$

$$= 0.39$$

$$\frac{63-\mu}{\sigma} = 1.43$$

$$63-\mu = 1.23\sigma \quad \text{——} ②$$

① − ② ⇒

$$-28 = \cancel{0.25\sigma} - 2.71\sigma$$

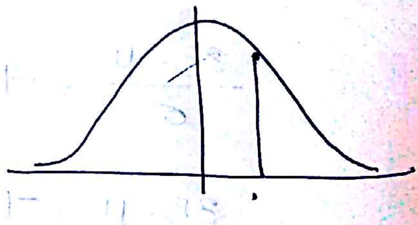
$$\cancel{\sigma = 112} \quad \sigma = \frac{-28}{-2.71} = 10.33$$

$$\mu = 63 - 1.23 \times 10.33$$

$$= 50.29$$

In a certain examination the % of candiates passing & getting distinctions were 45 & 9 resp. Evaluate the avg marks obtained by the candidate, the minimu pass & distinction marks being 40 & 75 resp. Assuming the dist to be normal.

$$P(X \geq 40) = 0.45 = \int_{\frac{40-\mu}{\sigma}}^{\infty} f(z)\,dz$$



$$0.45 = \int_0^{\infty} f(z)\,dz - \int_0^{\frac{40-\mu}{\sigma}} f(z)\,dz$$

$$0.45 = 0.5 - \int_0^{\frac{40-\mu}{\sigma}} f(z)\,dz$$

$$\int_0^{\frac{40-\mu}{\sigma}} = 0.5 - 0.45$$

$$= 0.05$$

$$\frac{40-\mu}{\sigma} = 0.13$$

$40-\mu = 0.13\sigma$ —— ①.

$$P(X \geq 75) = 0.09 = \int_{\frac{75-\mu}{\sigma}}^{\infty} f(z)dz$$

$$= \int_{0}^{\infty} f(z) - \int_{0}^{\frac{75-\mu}{\sigma}} f(z)$$

$$0.09 = 0.5 - \int_{0}^{\frac{75-\mu}{\sigma}}$$

$$\int_{0}^{\frac{75-\mu}{\sigma}} f(z)dz = 0.5 - 0.09$$

$$= 0.41$$

$75-\mu = 1.\cancel{4}34\sigma$ —— ②

$\mu = 36.27$

$\sigma = 28.6$

Q. If x is a normal variant with arithmetic mean 30 & sd 5, find $P(X=26)$.

Ans: $P(X=25) = P(25.5 \leq X \leq 26.5)$

$$= \int_{25.5}^{26.5} f(x)\,dx$$

~~f~~

For $x = 25.5$, $z = \frac{25.5-30}{5}$

$z =$

For $x = 26.5$, $z = \frac{26.5-30}{5} =$

$$\int_{25.5}^{26.5} f(x)\,dx = \int$$

=

=

Exponential distribution

Q. A CRV x having a pdf $f(x) = \lambda e^{-\lambda x}$ $x \geq 0$

$= 0$ otherwise

is said to be exponential distribution.

To find the mean & variance:

$$\text{Mean} = E(X) = \int_0^\infty x\, f(x)\,dx$$

$$= \int_0^\infty x \cdot \lambda e^{-\lambda x}\,dx$$

$$= \lambda \left\{ x \cdot \frac{e^{-\lambda x}}{-\lambda} - 1 \cdot \frac{e^{-\lambda x}}{\lambda^2} \right\}_0^\infty$$

$$= \lambda \left[0 + \frac{1}{\lambda^2} \right]$$

$$E(X) = \frac{1}{\lambda}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_0^{\infty} x^2 \lambda e^{-\lambda x}\, dx$$

$$= \lambda\left[x^2 \cdot \frac{e^{-\lambda x}}{-\lambda} - 2x \cdot \frac{e^{-\lambda x}}{\lambda^2} + 2 \cdot \frac{e^{-\lambda x}}{-\lambda^3}\right]_0^{\infty}$$

$$= \lambda\left[0 - \left(-\frac{2}{\lambda^3}\right)\right]$$

$$E(X^2) = +\frac{2}{\lambda^2}$$

$$V(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$V(X) = \frac{1}{\lambda^2}$$

Q. The time in hrs required to repair a machine x is exponentially dist. with parameter $\lambda = 1/20$. What is the probability that the required time (i) exceeds 30 hrs (ii) in b/w 16 & 24 hrs (iii) at most 10 hrs.

i) $P(X>30) = \int_{30}^{\infty} \frac{1}{20} e^{-x/20}\, dx$

=

ii) $P(16 \le x \le 24) = \int_{16}^{24} \frac{1}{20} e^{-x/20}\, dx$

=

iii) $P(X \le 10) = \int_0^{10} \frac{1}{20} e^{-x/20} dx$

$=$

09/03

Q. If X is an exp. random variable with parameter λ find ~~f(x)~~ dist. fn. of x.

Ans:
$$F(x) = P[X \le x]$$
$$= \int_{-\infty}^{x} f(x) dx$$
$$= \int_{-\infty}^{0} + \int_{0}^{x} f(x) dx$$
$$F(x) = 0 + \int_0^x \lambda e^{-\lambda x} dx$$
$$= \cancel{\lambda} \frac{e^{-\lambda x}}{-\cancel{\lambda}}$$
$$F(x) = \left[-e^{-\lambda x} \right]_0^x$$
$$= 1 - e^{-\lambda x}$$

$$F(x) = 0 \quad x < 0.$$
$$= 1 - e^{-\lambda x} \quad x \ge 0.$$

$$P(X > x) = 1 - P(X \le x)$$
$$= 1 - (1 - e^{-\lambda x})$$
$$= e^{-\lambda x}$$

i.e. if X is an exp. random variable

$P(X>x) = e^{-\lambda x}$

Let X be a random var. with pdf $f(x) = \frac{1}{3} e^{-x/3}$; $x>0$

$= 0$; o.w.

Find i) $P(X>3)$

ii) Mean & variance.

~~F(x)~~ i) $P(X>3) = 1 - P(X \leq 3)$

$= e^{-\lambda x}$

$= e^{-\frac{1}{3} \times 3}$ $(\lambda = 1/3,\ x = 3)$

$= \underline{0.3679}$

ii) $E(X) = \frac{1}{\lambda} = \frac{1}{1/3} = 3$ //.

$V(X) = \frac{1}{\lambda^2} = \frac{1}{1/9} = 9$ //

The time in hrs to repair a machine is exponentially dist with para $\lambda = 1/3$. What is prob that the repair time exceeds ~~i)~~ 3 hrs.

$P(X>3) = e^{-\lambda x}$

$= e^{-\frac{1}{3} \cdot 3}$

$= \underline{e^{-1}} = 0.36\underline{79}$

i.e. if x is an exp. dist. show ... given $X>s$ is equal to $P(X>t)$ for any $s,t>0$

Let X be an exp. random variable with parameter λ.

To show that x has the meomoryless ppty. for any $s,t>0$.

$$P(x>k)=\int_{k}^{\infty} f(x)\,dx$$

$$= e^{-\lambda k}$$

$$P(X>s+t)|X>s) = \frac{P(X>s+t \cap X>s)}{P(X>s)}$$

$$= \frac{P(X>s+t)}{P(X>s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t}$$

$$= P(X>t)$$

Q. Suppose a new machine is put into operation. Its life time is an exp random variable with mean life ~~10~~ hrs.

i) prob. that the machine will work continuously for 1 day

ii) suppose the machine has not failed by the end of the first day, prob that it will work for whole of ~~next first~~ day

FORMULAS OF FOURIER INTEGRAL & TRANSFORMS

(1) Fourier Integral representation of $f(x)$

$$f(x) = \int_0^\infty \left[A(w)\cos wx + B(w)\sin wx \right] dw$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv \, dv.$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv \, dv.$$

(2) Fourier cosine Integral of $f(x)$,

$$f(x) = \int_0^\infty A(w)\cos wx \, dw \quad , A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv \, dv$$

fourier sine integral of $f(x)$,

$$f(x) = \int_0^\infty B(w)\sin wx \, dw \quad , B(w) = \frac{2}{\pi}\int_0^\infty f(v)\sin wv \, dv.$$

(3) Fourier Transform of $f(x)$,

$$\hat{f}(w) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iwx} f(x)\, dx$$

Inverse fourier transform of $\hat{f}(w)$ is,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwx}\, dw$$

(4) Fourier cosine transform of $f(x)$,

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx \, dx$$

fourier sine transform of $f(x)$,

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx \, dx.$$

MODULE - III

FOURIER INTEGRALS AND TRANSFORMS

FOURIER INTEGRAL REPRESENTATION

$$f(x) = \int_0^\infty \left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right]$$

where
$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\, dv$$
$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\, dv$$

Q) Find the fourier integral representation of $f(x)$,

$$f(x) = \begin{cases} 0, & x<0 \\ 1/2, & x=0 \\ e^{-x}, & x>0 \end{cases}$$

Soln:-

Using fourier integral theorem,

$$f(x) = \int_0^\infty \left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right]d\omega \quad (1)$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\, dv$$

$$= \frac{1}{\pi}\int_0^\infty e^{-v}\cos\omega v\, dv$$

$$= \frac{1}{\pi}\left[\frac{-1}{1+\omega^2}\right] = -\frac{1}{\pi(1+\omega^2)}$$

Using the result,
$$\int_0^\infty e^{ax}\cos bx\, dx = \frac{a}{a^2+b^2}$$

$$\boxed{A(\omega) = \frac{-1}{\pi(1+\omega^2)}}$$

Now
$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\, dv$$

$$= \frac{1}{\pi}\int_0^\infty e^{-v}\sin\omega v\, dv$$

$$= \frac{1}{\pi}\left[\frac{\omega}{1+\omega^2}\right]$$

Using the result,
$$\int_0^\infty e^{ax}\sin bx\, dx = \frac{b}{a^2+b^2}$$

$$\Rightarrow \boxed{B(\omega) = \frac{\omega}{\pi(1+\omega^2)}}$$

Substituting $A(\omega)$ and $B(\omega)$ in ①

$$f(x) = \int_0^\infty \left[\frac{-1}{\pi(1+\omega^2)} \cos\omega x + \frac{\omega}{\pi(1+\omega^2)} \sin\omega x\right] d\omega$$

$$= \frac{1}{\pi} \int_0^\infty \left[\frac{-\cos\omega x + \omega \sin\omega x}{1+\omega^2}\right] d\omega = \begin{cases} 0, & x<0 \\ 1/2, & x=0 \\ e^{-x}, & x>0 \end{cases}$$

$$\Rightarrow \frac{1}{\pi}\int_0^\infty \left(\frac{-\cos\omega x + \omega\sin\omega x}{1+\omega^2}\right) d\omega = \begin{cases} 0, & x<0 \\ 1/2, & x=0 \\ e^{-x}, & x>0 \end{cases}$$

Q) Find the fourier integral representation of the function

$$f(x) = \begin{cases} 2 & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$$

Soln:

The fourier integral is,

$$f(x) = \int_0^\infty \left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right] d\omega$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) \cos\omega v \, dv$$

$$= \frac{1}{\pi}\int_{-1}^{1} 2 \cos\omega v \, dv = \frac{2}{\pi} \left[\frac{\sin\omega v}{\omega}\right]_{-1}^{1}$$

$$= \frac{2}{\pi}\left[\frac{\sin\omega - -\sin\omega}{\omega}\right] = \frac{2}{\pi}\left[\frac{2\sin\omega}{\omega}\right] = \frac{4}{\pi}\frac{\sin\omega}{\omega}$$

$$\Rightarrow \boxed{A(\omega) = \frac{4}{\pi}\frac{\sin\omega}{\omega}}$$

Now $B(\omega) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} f(v) \sin\omega v \, dv$

$$= \frac{1}{\pi}\int_{-1}^{1} 2 \sin\omega v \, dv$$

$\Rightarrow B(\omega) = \frac{2}{\pi}\int_{-1}^{1} \sin\omega v \, dv = \frac{2}{\pi}\left[\frac{-\cos\omega v}{\omega}\right]_{-1}^{1}$

$= \frac{2}{\pi}\left[\frac{-\cos\omega - -\cos(-\omega)}{\omega}\right] = \frac{2}{\pi}\left[\frac{-\cos\omega + \cos\omega}{\omega}\right] = \underline{\underline{0}}$

$\Rightarrow \boxed{B(\omega) = 0}$

$\therefore$ fourier integral, $f(x) = \int_0^\infty A(\omega)\cos\omega x + B(\omega)\sin\omega x \, d\omega$

$\Rightarrow f(x) = \int_0^\infty \left[\frac{4}{\pi}\frac{\sin\omega}{\omega}\cos\omega x + 0\right] d\omega$

$$f(x) = \underline{\underline{\frac{4}{\pi}\int_0^\infty \frac{\sin\omega \cos\omega x}{\omega} d\omega}}$$

Q) ~~Using fourier integral, prove that~~ $\cancel{e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \frac{(\omega^2+2)\cos\omega x}{\omega^2+4} d\omega}$

~~Soln:~~

Q) find the fourier integral of $f(x) = \begin{cases} \cos x, & 0<x<\pi \\ 0, & x>\pi \end{cases}$

Soln: fourier integral, $f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x] d\omega$

$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v \, dv$

$= \frac{1}{\pi}\int_0^\pi \cos v \cos\omega v \, dv = \frac{1}{\pi}\int_0^\pi \left[\frac{\cos(v+\omega v) + \cos(v-\omega v)}{2}\right] dv$

$= \frac{1}{2\pi}\int_0^\pi [\cos(1+\omega)v + \cos(1-\omega)v] \, dv$

$= \frac{1}{2\pi}\left\{\left[\frac{\sin(1+\omega)v}{1+\omega}\right]_0^\pi + \left[\frac{\sin(1-\omega)v}{1-\omega}\right]_0^\pi\right\}$

$= \frac{1}{2\pi}\left\{\frac{\sin(1+\omega)\pi - 0}{1+\omega} + \frac{\sin(1-\omega)\pi - 0}{1-\omega}\right\}$

$= \frac{1}{2\pi}\left\{\frac{\sin(\pi+\pi\omega)}{1+\omega} + \frac{\sin(\pi-\pi\omega)}{1-\omega}\right\}$

$$\Rightarrow A(\omega) = \frac{1}{2\pi}\left\{\frac{\bar{}\sin\pi\omega}{1+\omega} + \frac{\sin\pi\omega}{1-\omega}\right\}$$

$\sin(\pi+\theta) = \bar{}\sin\theta$

$\sin(\pi-\theta) = \sin\theta$

$\therefore \sin(\pi+\pi\omega) = \bar{}\sin\pi\omega$

$\sin(\pi-\pi\omega) = \sin\pi\omega$

$$= \frac{1}{2\pi}\left\{\frac{\sin\pi\omega}{1-\omega} + \frac{\bar{}\sin\pi\omega}{1+\omega}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{(1+\omega)\sin\pi\omega - (1-\omega)\sin\pi\omega}{(1+\omega)(1-\omega)}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{\sin\pi\omega + \omega\sin\pi\omega - \sin\pi\omega + \omega\sin\pi\omega}{(1-\omega^2)}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{2\omega\sin\pi\omega}{1-\omega^2}\right\} = \frac{1}{\pi}\left(\frac{\omega\sin\pi\omega}{1-\omega^2}\right)$$

$$\Rightarrow \boxed{A(\omega) = \frac{1}{\pi}\left(\frac{\omega\sin\pi\omega}{1-\omega^2}\right)}$$

Now $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\, dv$

$$B(\omega) = \frac{1}{\pi}\int_0^{\pi} \cos v \sin\omega v\, dv$$

$$= \frac{1}{\pi} * \int_0^{\pi}\left[\frac{\sin(v+\omega v) - \sin(v-\omega v)}{2}\right] dv$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\frac{\sin(1+\omega)v - \sin(1-\omega)v}{2}\right] dv$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\sin(1+\omega)v - \sin(1-\omega)v\right] dv$$

$$= \frac{1}{2\pi}\left\{\left[-\frac{\cos(1+\omega)v}{1+\omega}\right]_0^{\pi} + \left[\frac{\cos(1-\omega)v}{1-\omega}\right]_0^{\pi}\right\}$$

$$= \frac{1}{2\pi}\left\{-\left(\frac{\cos(\pi+\pi\omega) - \cos 0}{1+\omega}\right) + \left(\frac{\cos(\pi-\pi\omega) - \cos 0}{1-\omega}\right)\right\}$$

$$\Rightarrow B(\omega) = \frac{1}{2\pi}\left\{-\left(\frac{\bar{}\cos\pi\omega - 1}{1+\omega}\right) + \left(\frac{\bar{}\cos\pi\omega - 1}{1-\omega}\right)\right\}$$

$\boxed{\cos(\pi+\theta) = \bar{}\cos\theta}$
$\boxed{\cos(\pi-\theta) = \bar{}\cos\theta}$

$$\Rightarrow B(\omega) = \frac{1}{2\pi}\left\{\left(\frac{\cos\pi\omega+1}{1+\omega}\right) + \left(\frac{-\cos\pi\omega - 1}{1-\omega}\right)\right\}$$

$$\Rightarrow B(\omega) = \frac{1}{2\pi}\left\{\frac{(1-\omega)(\cos\pi\omega+1) + (-\cos\pi\omega-1)(1+\omega)}{(1+\omega)(1-\omega)}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{(\cos\pi\omega + 1 - \omega\cos\pi\omega) + (-\cos\pi\omega - \omega\cos\pi\omega - 1 - \omega)}{1-\omega^2}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{(\cos\pi\omega + 1 - \omega\cos\pi\omega - \omega) + (-\cos\pi\omega - \omega\cos\pi\omega - 1 - \omega)}{1-\omega^2}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{-2\omega\cos\pi\omega - 2\omega}{1-\omega^2}\right\} = \frac{1}{2\pi} \cdot \frac{-2\omega(\cos\pi\omega + 1)}{1-\omega^2}$$

$$\Rightarrow \boxed{B(\omega) = \frac{1}{\pi}\left(\frac{-\omega(\cos\omega\pi+1)}{1-\omega^2}\right)}$$

$\therefore$ fourier integral of $f(x)$ is,

$$f(x) = \int_0^\infty \left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right] d\omega$$

$$= \int_0^\infty \left[\frac{1}{\pi}\left(\frac{\omega\sin\pi\omega}{1-\omega^2}\right)\cos\omega x + \frac{1}{\pi}\left(\frac{-\omega(\cos\omega\pi+1)}{1-\omega^2}\right)\sin\omega x\right] d\omega$$

$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^\infty \left[\frac{\omega\sin\pi\omega\cos\omega x - \omega(\cos\omega\pi+1)\sin\omega x}{1-\omega^2}\right] d\omega$$

Q) Find the fourier integral representation of the function

$$f(x) = \begin{cases} 1 & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$$

Soln:

fourier integral representation, of $f(x)$

$$f(x) = \int_0^\infty A(\omega)\cos\omega x + B(\omega)\sin\omega x \quad d\omega$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\, dv$$

$$= \frac{1}{\pi}\int_{-1}^{1} 1.\cos\omega v\, dv = \frac{1}{\pi}\left[\frac{\sin\omega v}{\omega}\right]_{-1}^{1}$$

$$= \frac{1}{\pi}\left[\frac{\sin\omega - {}^{-}\sin\omega}{\omega}\right] \qquad \because \sin(-\omega) = -\sin\omega$$

$$= \frac{1}{\pi}\left[\frac{2\sin\omega}{\omega}\right] = \frac{2}{\pi}\left[\frac{\sin\omega}{\omega}\right]$$

$$\Rightarrow \boxed{A(\omega) = \frac{2}{\pi}\frac{\sin\omega}{\omega}}$$

Now $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\, dv$

$$= \frac{1}{\pi}\int_{-1}^{1} 1.\sin\omega v\, dv = \frac{1}{\pi}\left[\frac{-\cos\omega v}{\omega}\right]_{-1}^{1}$$

$$= \frac{1}{\pi}\left[\frac{\cos\omega - \cos(\omega)}{\omega}\right] = \frac{1}{\pi}\left(\frac{\cos\omega - \cos\omega}{\omega}\right) \qquad \because \cos(-\omega) = \cos\omega$$

$$= 0$$

$$\Rightarrow \boxed{B(\omega) = 0}$$

$\therefore$ fourier integral, $f(x) = \int_0^{\infty}\left(A(\omega)\cos\omega x + B(\omega)\sin\omega x\right) d\omega$

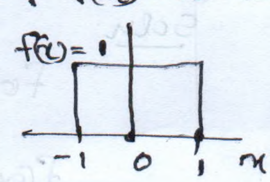
$$= \int_0^{\infty}\left[\frac{2}{\pi}\frac{\sin\omega}{\omega}\cos\omega x + 0\right] d\omega$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{\sin\omega\cos\omega x}{\omega}\, d\omega = \begin{cases} 1 & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$$

$$\Rightarrow \int_0^{\infty}\frac{\sin\omega\cos\omega x}{\omega}\, d\omega = \begin{cases} \frac{\pi}{2} & \text{if } 0<x<1 \\ 0 & \text{if } x>1 \end{cases}$$

when $x=1$, using fourier integral theorem, ~~fourier~~ fourier integral is average of left and right hand limits of $f(x)$ at $x=1$.

$$\Rightarrow \frac{(1+0)}{2} = \frac{1}{2}$$



$$\Rightarrow \int_0^{\infty}\frac{\sin\omega\cos\omega x}{\omega}\, d\omega = \frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi}{4} \;//$$

FOURIER SINE INTEGRAL AND COSINE INTEGRALS

Fourier Sine Integral :-

$$f(x) = \int_0^\infty B(w) \sin wx \, dw$$

where $B(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv$

Fourier Cosine Integral :-

$$f(x) = \int_0^\infty A(w) \cos wx \, dw$$

$$A(w) = \frac{2}{\pi} \int_0^\infty f(v) \cos wv \, dv$$

Q) Find the fourier sine and cosine integral of

$$f(x) = \begin{cases} \sin x, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$$

Soln:

Fourier Sine integral, $f(x) = \int_0^\infty B(w) \sin wx \, dw$

where $B(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv$

$$\Rightarrow B(w) = \frac{2}{\pi} \int_0^\pi \sin v \sin wv \, dv$$

$$= \frac{2}{\pi} \int_0^\pi \left[\frac{\cos(v - wv) - \cos(v + wv)}{2} \right] dv$$

$$= \frac{2}{2 \cdot \pi} \int_0^\pi \left[\cos(1-w)v - \cos(1+w)v \right] dv$$

$$= \frac{1}{\pi} \left\{ \left[\frac{\sin(1-w)v}{1-w} \right]_0^\pi - \left[\frac{\sin(1+w)v}{1+w} \right]_0^\pi \right\}$$

$$= \frac{1}{\pi} \left\{ \left[\frac{\sin(\pi - \pi w)}{1-w} - 0 \right] - \left[\frac{\sin(\pi + \pi w)}{1+w} - 0 \right] \right\}$$

$$\Rightarrow B(w) = \frac{1}{\pi}\left[\frac{\sin\pi w}{1-w} - \frac{-\sin\pi w}{1+w}\right] \qquad \because \sin(\pi+\theta) = -\sin\theta, \quad \sin(\pi-\theta) = \sin\theta$$

$$= \frac{1}{\pi}\left[\frac{\sin\pi w}{1-w} + \frac{\sin\pi w}{1+w}\right]$$

$$= \frac{1}{\pi}\left[\frac{(1+w)\sin\pi w + (1-w)\sin\pi w}{(1-w)(1+w)}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin\pi w + w\sin\pi w + \sin\pi w - w\sin\pi w}{1-w^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\sin\pi w}{1-w^2}\right] = \frac{2}{\pi}\left(\frac{\sin\pi w}{1-w^2}\right)$$

$$\Rightarrow \boxed{B(w) = \frac{2}{\pi}\,\frac{\sin\pi w}{1-w^2}}$$

$\therefore$ fourier sine integral $f(x) = \int_0^\infty B(w)\sin wx\,dw$

$$\Rightarrow f(x) = \int_0^\infty \frac{2}{\pi}\,\frac{\sin\pi w}{1-w^2}\,\sin wx\,dw$$

$$\Rightarrow \boxed{f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\pi w}{1-w^2}\,\sin wx\,dw}$$

Now fourier Cosine integral is

$$f(x) = \int_0^\infty A(w)\cos wx\,dw$$

where $A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\,dv$

$$\Rightarrow A(w) = \frac{2}{\pi}\int_0^\pi \sin v\cos wv\,dv$$

$$= \frac{2}{\pi}\int_0^\pi\left[\frac{\sin(v+wv) + \sin(v-wv)}{2}\right]dv$$

$$\Rightarrow A(w) = \frac{1}{\pi}\int_0^\pi\left[\sin(1+w)v + \sin(1-w)v\right]dv$$

$$= \frac{1}{\pi}\left\{\left[\frac{-\cos(1+w)v}{1+w}\right]_0^\pi + \left[\frac{-\cos(1-w)v}{1-w}\right]_0^\pi\right.$$

$$\Rightarrow A(w) = -\frac{1}{\pi}\left\{\left[\frac{\cos(1+w)x}{1+w}\right]_0^{\pi} + \left[\frac{\cos(1-w)x}{1-w}\right]_0^{\pi}\right\}$$

$$A(w) = -\frac{1}{\pi}\left\{\left(\frac{\cos(\pi+w\pi)-\cos 0}{1+w}\right) + \left(\frac{\cos(\pi-w\pi)-\cos 0}{1-w}\right)\right\}$$

$$A(w) = -\frac{1}{\pi}\left\{\frac{-\cos w\pi - 1}{1+w} + \frac{-\cos w\pi - 1}{1-w}\right\}$$

$$A(w) = -\frac{1}{\pi}\left\{\frac{-(\cos w\pi+1)}{1+w} + \frac{-(\cos w\pi+1)}{1-w}\right\} \qquad \because \cos(\pi+\theta) = -\cos\theta, \ \cos(\pi-\theta) = -\cos\theta$$

$$A(w) = \frac{1}{\pi}\left\{\frac{\cos w\pi+1}{1+w} + \frac{\cos w\pi+1}{1-w}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{(1-w)(\cos w\pi+1) + (1+w)(\cos w\pi+1)}{(1+w)(1-w)}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{(\cos w\pi+1-w\cos w\pi - w) + (\cos w\pi+1+w\cos w\pi+w)}{1-w^2}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{2\cos w\pi+2}{1-w^2}\right\} = \frac{2}{\pi}\frac{(\cos w\pi+1)}{1-w^2}$$

$$\Rightarrow \boxed{A(w) = \frac{2}{\pi}\frac{(\cos w\pi+1)}{1-w^2}}$$

$\therefore$ Fourier sine integral is

$$f(x) = \int_0^{\infty} A(w)\sin wx \, dw$$

$$= \int_0^{\infty} \frac{2}{\pi}\frac{(\cos w\pi+1)}{1-w^2}\sin wx \, dw$$

$$\boxed{f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{(\cos w\pi+1)}{1-w^2}\sin wx \, dw}$$

Q) Represent $f(x) = \begin{cases} x^2, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$ as a fourier cosine integral.

Soln.

fourier sine integral is,

$$f(x) = \int_0^\infty B(w) \sin wx \, dw \quad , \quad B(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv$$

$$B(w) = \frac{2}{\pi} \int_0^1 v^2 \sin wv \, dv$$

$$= \frac{2}{\pi} \left\{ \left[\left(v^2 \cdot \frac{-\cos wv}{w} \right)_0^1 - \int_0^1 2v \frac{-\cos wv}{w} dv \right\}$$

$$= \frac{2}{\pi} \left\{ -\left(1 \cdot \frac{\cos w}{w} - 0\right) + \frac{2}{w} \int_0^1 v \cos wv \, dv \right\}$$

$$= \frac{2}{\pi} \left\{ -\frac{\cos w}{w} + \frac{2}{w} \left\{ \left[v \frac{\sin wv}{w} \right]_0^1 - \int_0^1 1 \cdot \frac{\sin wv}{w} dv \right. \right.$$

$$= \frac{2}{\pi} \left\{ -\frac{\cos w}{w} + \frac{2}{w} \left\{ \left[1 \cdot \frac{\sin w}{w} - \left[\frac{-\cos wv}{w^2} \right]_0^1 \right\} \right.$$

$$= \frac{2}{\pi} \left\{ -\frac{\cos w}{w} + \frac{2}{w} \left\{ \frac{\sin w}{w} + \frac{\cos w - 1}{w^2} \right\} \right\}$$

$$B(w) = \frac{2}{\pi} \left\{ -\frac{\cos w}{w} + \frac{2 \sin w}{w^2} + \frac{2(\cos w - 1)}{w^3} \right\}$$

$\therefore$ fourier sine integral is,

$$f(x) = \int_0^\infty B(w) \sin wx \, dw$$

$$= \int_0^\infty \frac{2}{\pi} \left\{ -\frac{\cos w}{w} + \frac{2 \sin w}{w^2} + \frac{2(\cos w - 1)}{w^3} \right\} \sin wx \, dw$$

Q) Find fourier cosine integral of $f(x) = \begin{cases} e^{-x}, & \text{if } 0<x<a \\ 0 & \text{if } x>a \end{cases}$

Soln:

fourier sine integral is,

$$f(x) = \frac{2}{\pi}\int_0^\infty B(w)\, \sin wx \, dw$$

where $B(w) = \frac{2}{\pi}\int_0^\infty f(u) \sin wu \, du$

$$= \frac{2}{\pi}\int_0^a e^{-u} \sin wu \, du$$

$$= \frac{2}{\pi}\int_0^a e^{-u} \sin wu \, du$$

$$= \frac{2}{\pi}\left\{\left[\frac{e^{-u}}{1+w^2}\left(-\sin wx - w\cos wx\right)\right]_0^a\right\}$$

$$= \frac{2}{\pi}\left\{\frac{e^{-a}}{1+w^2}\left(-\sin wa - w\cos wa\right) - \frac{1}{1+w^2}\left(-w\cdot\cos 0\right)\right\}$$

$$= \frac{2}{\pi}\left\{\frac{e^{-a}}{1+w^2}\left(-\sin wa - w\cos wa\right) + \frac{w}{1+w^2}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{e^{-a}(-\sin wa - w\cos wa) + w}{1+w^2}\right\}$$

$$\therefore f(x) = \int_0^\infty B(w) \sin wx \, dw$$

$$= \int_0^\infty \frac{2}{\pi}\left\{\frac{e^{-a}(-\sin wa - w\cos wa) + w}{1+w^2}\right\} \sin wx \, dw$$

Q) Find fourier sine integral of $f(x) = \begin{cases} x & \text{if } 0<x<a \\ 0 & \text{if } x>a \end{cases}$

Soln:

fourier sine integral of $f(x)$,

$$f(x) = \int_0^\infty B(w) \sin wx \, dw$$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin\omega v\,dv$$

$$= \frac{2}{\pi}\int_0^a v\sin\omega v\,dv$$

$$= \frac{2}{\pi}\left\{\left[v\cdot\frac{-\cos\omega v}{\omega}\right]_0^a - \int_0^a 1\cdot\frac{-\cos\omega v}{\omega}\,dv\right.$$

$$= \frac{2}{\pi}\left\{-\left(\frac{a\cos\omega a}{\omega}-0\right) + \left[\frac{\sin\omega v}{\omega^2}\right]_0^a\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-a\cos\omega a}{\omega} + \frac{\sin\omega a}{\omega^2}\right\}$$

$$B(\omega) = \frac{2}{\pi}\left\{\frac{-a\omega\cos\omega a + \sin\omega a}{\omega^2}\right\}$$

$\therefore$ fourier sine integral is,

$$f(x) = \int_0^\infty B(\omega)\sin\omega x\,d\omega$$

$$\Rightarrow f(x) = \int_0^\infty \frac{2}{\pi}\left\{\frac{-a\omega\cos\omega a+\sin\omega a}{\omega^2}\right\}\sin\omega x\,d\omega$$

Basic Results

① $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

② $\int_0^\infty e^{ax}\sin bx\,dx = \frac{b}{a^2+b^2}$

③ $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

④ $\int_0^\infty e^{ax}\cos bx\,dx = \frac{a}{a^2+b^2}$

FOURIER TRANSFORM

fourier transform of $f(x)$ is,

$$f^{\wedge}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

INVERSE FOURIER TRANSFORM

Inverse fourier transform of $f^{\wedge}(w)$ is,

$$F^{-1}(f^{\wedge}(w)) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^{\wedge}(w)\, e^{iwx}\, dw$$

Another notation for fourier transform is, $\hat{f} = \mathcal{F}(f)$

and another notation for Inverse transform is $\mathcal{F}^{-1}(\hat{f}) = f$

Q) Find the fourier transform of $f(x) = \begin{cases} 1 & \text{if } |x|<1 \\ 0 & \text{otherwise} \end{cases}$

Soln:

Fourier transform of $f(x)$,

$$f^{\wedge}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1 \cdot e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-iwx}}{-iw} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-iw} - e^{iw}}{-iw} \right] = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{iw} - e^{-iw}}{iw} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2i \sin w}{iw} \right]$$

Result:

$$\frac{e^{iw} - e^{-iw}}{2i} = \sin w$$

$$\frac{e^{iw} + e^{-iw}}{2} = \cos w$$

$$= \frac{2}{\sqrt{2\pi}} \frac{\sin w}{w}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sin w}{w}$$

$$\therefore f^{\wedge}(w) = \sqrt{\frac{2}{\pi}} \frac{\sin w}{w}$$

Q) Find the fourier transform $\hat{f}(e^{-ax})$ of $f(x) = \begin{cases} e^{-ax}, & x>0 \\ 0, & x\le 0 \end{cases}$

Soln:

Fourier transform of $f(x)$ is,

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{-i\omega x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} e^{-i\omega x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x(a+i\omega)} \, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-x(a+i\omega)}}{-(a+i\omega)} \right]_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{0 - e^{0}}{-(a+i\omega)} \right] = \frac{1}{\sqrt{2\pi}} \frac{1}{(a+i\omega)}$$

$$\Rightarrow \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{(a+i\omega)}$$

Q) i) Find the fourier transform of $f(x)$ where

$$f(x) = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

ii) Using inverse fourier transform show that

$$\int_{0}^{\infty} \cos\left(\frac{\omega}{2}\right) \frac{\sin\omega - \omega\cos\omega}{\omega^3} \, d\omega = \frac{3\pi}{16}$$

Soln:-

Fourier transform of $f(x)$, $F(f(x)) = \hat{f}(\omega)$

$$\Rightarrow \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2) e^{-i\omega x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)(\cos\omega x - i\sin\omega x) \, dx$$

$\because e^{-ix} = \cos x - i\sin x$

$e^{-i\omega x} = \cos\omega x - i\sin\omega x$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2) \cos\omega x \, dx$$

Since $\int_{-1}^{1} (1-x^2) \sin\omega x \, dx = 0$ being odd function

$$\Rightarrow \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2) \cos \omega x \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \, 2 \int_0^1 (1-x^2) \cos \omega x \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ \left((1-x^2) \frac{\sin \omega x}{\omega} \right)_0^1 - \int_0^1 -2x \frac{\sin \omega x}{\omega} dx \right\}$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ 0 + \frac{2}{\omega} \int_0^1 x \sin \omega x \, dx \right\}$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ \frac{2}{\omega} \left\{ \left(x \cdot \frac{-\cos \omega x}{\omega} \right)_0^1 - \int_0^1 \frac{-\cos \omega x}{\omega} dx \right.\right.$$

$$= \frac{4}{\sqrt{2\pi}\,\omega} \left\{ \left(\frac{-1 \cdot \cos \omega}{\omega} - 0 \right) + \frac{1}{\omega} \left(\frac{\sin \omega x}{\omega} \right)_0^1 \right.$$

$$= \frac{4}{\sqrt{2\pi}\,\omega} \left\{ \frac{-\cos \omega}{\omega} + \frac{1}{\omega^2} (\sin \omega - 0) \right\}$$

$$= \frac{4}{\sqrt{2\pi}} \left\{ \frac{-\cos \omega}{\omega^2} + \frac{\sin \omega}{\omega^3} \right\} = \frac{4}{\sqrt{2\pi}} \left\{ \frac{-\omega \cos \omega + \sin \omega}{\omega^3} \right\}$$

$$\hat{f}(\omega) = \frac{4}{\sqrt{2\pi}} \left[\frac{\sin \omega - \omega \cos \omega}{\omega^3} \right]$$

ii) Taking inverse fourier transform of $\hat{f}(\omega)$, we've

$$f(x) = F^{-1}(\hat{f}(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{4}{\sqrt{2\pi}} \left[\frac{\sin \omega - \omega \cos \omega}{\omega^3} \right] e^{i\omega x} d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \times \frac{4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{\sin \omega - \omega \cos \omega}{\omega^3} \right] (\cos \omega x + i \sin \omega x) \, d\omega$$

$$\Rightarrow f(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin \omega - \omega \cos \omega}{\omega^3} \right) \cos \omega x \, d\omega = f(x)$$

Result :- $e^{i\omega x} = \cos \omega x + i \sin \omega x$

$$\left[\text{Since} \int_{-\infty}^{\infty} \left(\frac{\sin \omega - \omega \cos \omega}{\omega^3} \right) \sin \omega x \, d\omega = 0 \right]$$

This gives, $\frac{2}{\pi} \times 2\int_0^\infty \left[\frac{\sin w - w\cos w}{w^3}\right] \cos wx \, dw = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|>1 \end{cases}$

$$\Rightarrow \frac{4}{\pi}\int_0^\infty \left[\frac{\sin w - w\cos w}{w^3}\right] \cos wx \, dw = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

when $x = \frac{1}{2}$, we get

$$\frac{4}{\pi}\int_0^\infty \left[\frac{\sin w - w\cos w}{w^3}\right] \cos \frac{w}{2} \, dw = \left\{ 1 - \left(\frac{1}{2}\right)^2, \quad |x|<1 \right.$$

$$\Rightarrow \frac{4}{\pi}\int_0^\infty \left[\frac{\sin w - w\cos w}{w^3}\right] \cos \frac{w}{2} \, dw = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\Rightarrow \int_0^\infty \left[\frac{\sin w - w\cos w}{w^3}\right] \cos \frac{w}{2} \, dw = \frac{\pi}{4} \cdot \frac{3}{4}$$

$$\Rightarrow \int_0^\infty \left[\frac{\sin w - w\cos w}{w^3}\right] \cos \frac{w}{2} \, dw = \frac{3\pi}{16}$$

Hence the proof.

Q) Find the fourier transform of e^{-x^2}.

Soln:

fourier transform of $f(x)$,

$$f^\wedge(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^{-iwx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2+iwx)} \, dx$$

$$\Rightarrow f^\wedge(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x+\frac{iw}{2}\right)^2} e^{-\frac{w^2}{4}} \, dx$$

write $(x^2+iwx) = \left(x+\frac{iw}{2}\right)^2 - \frac{w^2}{4}$

put $\left(x + \frac{iw}{2}\right) = t \Rightarrow dt = dx$

$$\Rightarrow f^\wedge(w) = \frac{e^{-\frac{w^2}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \, dt = \frac{e^{-w^2/4}}{\sqrt{2\pi}} \left[\sqrt{\pi}\right]$$

$\because \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$

$$\therefore\ f^{\wedge}(\omega) = \frac{1}{\sqrt{2}} e^{-\omega^2/4}$$

Q) Find the fourier transform of

$$f(x) = \left\{ e^{-|x|},\ -\infty < x < \infty \right.$$

Soln:

fourier transform of $f(x) = \cancel{f}\ f^{\wedge}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x} dx$

$$\Rightarrow f^{\wedge}(\omega) = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} e^{x} e^{-i\omega x} dx + \int_{0}^{\infty} e^{-x} e^{-i\omega x} dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} e^{x(1-i\omega)} dx + \int_{0}^{\infty} e^{-x(1+i\omega)} dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[\frac{e^{x(1-i\omega)}}{1-i\omega} \right]_{-\infty}^{0} + \left[\frac{e^{-x(1+i\omega)}}{-(1+i\omega)} \right]_{0}^{\infty} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{(e^{0} - e^{-\infty})}{1-i\omega} + \frac{(e^{-\infty} - e^{0})}{-(1+i\omega)} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1-i\omega} + \frac{(0-1)}{-(1+i\omega)} \right] = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1-i\omega} + \frac{1}{1+i\omega} \right]$$

$$f^{\wedge}(\omega) = \frac{1}{\sqrt{2\pi}} \left[\frac{1+i\omega + 1-i\omega}{(1-i\omega)(1+i\omega)} \right] = \frac{1}{\sqrt{2\pi}} \left[\frac{2}{1^2-(i\omega)^2} \right] = \frac{1}{\sqrt{2\pi}} \left(\frac{2}{1+\omega^2} \right)$$

$$\Rightarrow \boxed{f^{\wedge}(\omega) = \frac{2}{\sqrt{2\pi}} \left(\frac{1}{1+\omega^2} \right) = \sqrt{\frac{2}{\pi}}\, \frac{1}{1+\omega^2}}$$

Q) Find the fourier transform of $f(x) = \begin{cases} |x| & \text{if } -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$

Soln:-

fourier transform of $f(x)$,

$$f^{\wedge}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x} dx.$$

$$\Rightarrow f^{\wedge}(\omega) = \frac{1}{\sqrt{2\pi}}\left\{\int_{-1}^{1} |x| . e^{-i\omega x} dx\right\}$$

$$\Rightarrow f^{\wedge}(\omega) = \frac{1}{\sqrt{2\pi}}\left\{\int_{-1}^{0} -x . e^{-i\omega x} dx + \int_{0}^{1} x e^{-i\omega x} dx\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{-\left[\left(\frac{x . e^{-i\omega x}}{-i\omega}\right)_{-1}^{0} - \int_{-1}^{0} \frac{1 . e^{-i\omega x}}{-i\omega} dx\right] + \left[\left(\frac{x . e^{-i\omega x}}{-i\omega}\right)_{0}^{1} - \int_{0}^{1} \frac{1 . e^{-i\omega x}}{-i\omega} dx\right]\right\}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\left\{-\left[\left(\frac{0 - -1 . e^{i\omega}}{-i\omega}\right] + \frac{-1}{i\omega}\left[\frac{e^{-i\omega x}}{-i\omega}\right]_{-1}^{0} + \left[\frac{1 . e^{-i\omega} - 0}{-i\omega}\right] + \frac{-1}{i\omega}\left[\frac{e^{-i\omega x}}{-i\omega}\right]_{0}^{1}\right\}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\left\{\left(+\frac{e^{+i\omega}}{i\omega}\right) + \frac{-1}{(i\omega)^2}\left(e^{0} - e^{i\omega}\right) + -\left[\frac{e^{-i\omega}}{i\omega}\right] - \frac{1}{(i\omega)^2}\left[e^{-i\omega} - e^{0}\right]\right\}$$

$$\Rightarrow f^{\wedge}(\omega) = \frac{1}{\sqrt{2\pi}}\left\{\frac{+e^{+i\omega}}{i\omega} + \frac{-(1 - e^{i\omega})}{(i\omega)^2} - \frac{e^{-i\omega}}{i\omega} - \frac{e^{-i\omega}}{(i\omega)^2} + \frac{1}{(i\omega)^2}\right\}$$

$$\cancel{= \frac{1}{\sqrt{2\pi}}\left\{-\frac{2e^{-i\omega}}{i\omega} + 2.\right.}$$

$$\Rightarrow f^{\wedge}(\omega) = \frac{1}{\sqrt{2\pi}}\left\{\left(\frac{e^{i\omega}}{i\omega} - \frac{e^{-i\omega}}{i\omega}\right) + \left(\frac{+e^{i\omega}}{(i\omega)^2} - \frac{e^{-i\omega}}{(i\omega)^2}\right) - \left(\frac{e^{-i\omega}}{i\omega}\right)\right\}.$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left(\frac{e^{i\omega} - e^{-i\omega}}{i\omega}\right) + \left(\frac{e^{i\omega} - e^{-i\omega}}{(i\omega)^2}\right) - \left(\frac{e^{-i\omega}}{i\omega}\right)\right\}$$

$$\Rightarrow f^{\wedge}(\omega) = \frac{1}{\sqrt{2\pi}}\left\{\frac{2\sin\omega}{\omega} + \frac{2i\sin\omega}{-\omega^2} - \frac{e^{-i\omega}}{i\omega}\right\}.$$

$$\Rightarrow f^{\wedge}(\omega) = \frac{1}{\sqrt{2\pi}}\left\{\frac{2\sin\omega}{\omega} - \frac{2i\sin\omega}{\omega^2} - \frac{e^{-i\omega}}{i\omega}\right\}$$

Using the result,

$e^{i\omega} + e^{-i\omega} = 2\cos\omega$

$e^{i\omega} - e^{-i\omega} = 2i\sin\omega$

FOURIER COSINE AND SINE TRANSFORMS

Fourier cosine transform of $f(x)$ is,

$$f_c^{\wedge}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos\omega x \, dx$$

Inverse fourier cosine transform of $f_c^{\wedge}(\omega)$ is,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c^{\wedge}(\omega) \cos\omega x \, d\omega.$$

fourier sine transform of $f(x)$ is,

$$f_s^{\wedge}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin\omega x \, dx.$$

Inverse fourier sine transform of $f_s^{\wedge}(\omega)$ is,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s^{\wedge}(\omega) \sin\omega x \, d\omega.$$

Q) Find fourier sine and cosine transforms of the function

$$f(x) = \begin{cases} k & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$$

soln:

fourier cosine transform of $f(x)$

$$f_c^{\wedge}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{a} k \cos\omega x \, dx = \sqrt{\frac{2}{\pi}}\, k \int_0^{a} \cos\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}}\, k \left[\frac{\sin\omega x}{\omega}\right]_0^a = \sqrt{\frac{2}{\pi}}\, k \left[\frac{\sin\omega a}{\omega} - 0\right]$$

$$\Rightarrow \boxed{f_c^{\wedge}(\omega) = \sqrt{\frac{2}{\pi}}\, k \, \frac{\sin\omega a}{\omega}}$$

fourier sine transform of $f(x)$,

$$f_s^{\wedge}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{a} k \cdot \sin\omega x \, dx$$

$$\Rightarrow f_s^{\wedge}(w) = \sqrt{\frac{2}{\pi}}k\int_0^a \sin wx\, dx = \sqrt{\frac{2}{\pi}}k\left[\frac{-\cos wx}{w}\right]_0^a$$

$$\Rightarrow f_s^{\wedge}(w) = \sqrt{\frac{2}{\pi}}k\left[\frac{-(\cos wa - 1)}{w}\right] = \sqrt{\frac{2}{\pi}}k\left[\frac{1-\cos wa}{w}\right]$$

$$\Rightarrow \boxed{f_s^{\wedge}(w) = \sqrt{\frac{2}{\pi}}k\left[\frac{1-\cos wa}{w}\right]}$$

Q) Find the fourier cosine transform of exponential function.

Soln:

Exponential function, $f(x) = \begin{cases} e^{-x}, & x>0 \\ 0 & \text{otherwise.} \end{cases}$

fourier cosine transform of $f(x)$,

$$f_c^{\wedge}(w) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos wx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}\cos wx\, dx$$

$$\Rightarrow f_c^{\wedge}(w) = \sqrt{\frac{2}{\pi}}\left[\frac{-1}{1+w^2}\right] = \frac{-\sqrt{\frac{2}{\pi}}}{1+w^2}$$

Using the result,

$$\boxed{\int_0^{\infty} e^{ax}\cos bx\, dx = \frac{a}{a^2+b^2}} \quad — (a)$$

Q) Find the Fourier cosine and sine transform of $f(x) = e^{-ax}$

Soln:

fourier cosine transform of $f(x)$ is,

$$f_c^{\wedge}(w) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos wx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\cos wx\, dx$$

$$\Rightarrow f_c^{\wedge}(w) = \sqrt{\frac{2}{\pi}}\left[\frac{-a}{-a^2+w^2}\right] = \sqrt{\frac{2}{\pi}}\left(\frac{-a}{a^2+w^2}\right) \quad \text{Using result (a)}$$

Fourier sine transform of $f(x)$,

$$f_s^\wedge(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin\omega x \, dx$$

$$\Rightarrow f_s^\wedge(\omega) = \sqrt{\frac{2}{\pi}} \left[\frac{\omega}{a^2+\omega^2} \right] //$$

$$\boxed{\because \text{ using the result, } \int_0^\infty e^{ax} \sin bx \, dx = \frac{b}{a^2+b^2}}$$

a) Find the fourier sine transform of $e^{-|x|}$. Hence evaluate $\int_0^\infty \frac{\omega \sin\omega x}{1+\omega^2} dx$.

Soln:

$$f(x) = e^{-|x|} \Rightarrow f(x) = e^{-x}, \; 0 < x < \infty$$

$\therefore$ fourier sine transform of $f(x)$ is,

$$f_s^\wedge(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cdot \sin\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\omega}{(-1)^2+\omega^2} \right]$$

$$\Rightarrow f_s^\wedge(\omega) = \sqrt{\frac{2}{\pi}} \left[\frac{\omega}{1+\omega^2} \right]$$

$$\boxed{\because \text{ using the result } \int_0^\infty e^{ax} \sin bx \, dx = \frac{b}{a^2+b^2}}$$

Using inverse fourier sine transform, we get

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s^\wedge(\omega) \sin\omega x \, d\omega$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{\omega}{1+\omega^2} \right) \sin\omega x \, d\omega$$

$$\Rightarrow f(x) = \sqrt{\frac{2}{\pi}} \times \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\omega \sin\omega x}{1+\omega^2} d\omega$$

ie $f(x) = e^{-x} = \sqrt{\frac{2}{\pi}} \times \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{w \sin wx}{1+w^2}\, dw$

$$\Rightarrow e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{w \sin wx}{1+w^2}\, dw \qquad \because f(x) = e^{-x}$$

$$\Rightarrow \underline{\underline{\int_0^\infty \frac{w \sin wx}{1+w^2}\, dw = \frac{\pi}{2} e^{-x}}}$$ Hence the proof.

Q) Find the fourier sine transform of

$$f(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>0 \end{cases}$$

Soln:

The fourier sine transform is given by,

$$f_s^\wedge(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left[\int_0^1 x \sin wx \, dx \right] + \left[\int_1^2 (2-x) \sin wx \right] dx \right.$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \sin wx \, dx + \int_1^2 (2-x) \sin wx \, dx \right.$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \left\{ \left[\left(x \cdot \frac{-\cos wx}{w} \right)_0^1 - \int_0^1 \frac{-\cos wx}{w} dx \right] + \left[\left((2-x) \cdot \frac{-\cos wx}{w} \right)_1^2 - \int_1^2 1 \cdot \frac{-\cos wx}{w} \right. \right.$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \left\{ \left[\left[-\left(1 \cdot \frac{\cos w}{w} - 0\right) \right] + \left[\frac{\sin wx}{w^2} \right]_0^1 \right] + \left[-\left(0 - \frac{\cos w}{w}\right) - \left[\frac{\sin wx}{w^2} \right]_1^2 \right] \right\}$$

$$\Rightarrow f_s^\wedge(w) = \sqrt{\frac{2}{\pi}} \left\{ \left[\frac{-\cos w}{w} + \frac{\sin w - 0}{w^2} \right] + \left[\frac{\cos w}{w} - \left(\frac{\sin 2w - \sin w}{w^2} \right) \right] \right\}$$

$$\Rightarrow f_s^\wedge(w) = \sqrt{\frac{2}{\pi}} \left\{ \cancel{\frac{-\cos w}{w}} + \frac{\sin w}{w^2} + \cancel{\frac{\cos w}{w}} - \frac{\sin 2w}{w^2} + \frac{\sin w}{w^2} \right\}$$

$$\Rightarrow f_s^\wedge(w) = \sqrt{\frac{2}{\pi}} \left\{ \frac{2\sin w}{w^2} - \frac{\sin 2w}{w^2} \right\}$$

$$\Rightarrow \boxed{f_s^{\wedge}(\omega) = \sqrt{\frac{2}{\pi}}\left[\frac{2\sin\omega - \sin 2\omega}{\omega^2}\right]}$$

Q) Find the fourier cosine transform $f_c^{\wedge}(\omega)$ of

$$f(x) = \begin{cases} 1 & \text{if } 0<x<1 \\ -1 & \text{if } 1<x<2 \\ 0 & \text{if } x>2. \end{cases}$$

Soln:-

Fourier cosine transform of $f(x)$ is,

$$f_c^{\wedge}(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\omega x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left\{\int_0^1 1.\cos\omega x\, dx + \int_1^2 -1.\cos\omega x\, dx\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left[\frac{\sin\omega x}{\omega}\right]_0^1 - \left[\frac{\sin\omega x}{\omega}\right]_1^2\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{\frac{\sin\omega}{\omega} - \left(\frac{\sin 2\omega - \sin\omega}{\omega}\right)\right\}$$

$$\Rightarrow \boxed{f_c^{\wedge}(\omega) = \sqrt{\frac{2}{\pi}}\left\{\frac{2\sin\omega - \sin 2\omega}{\omega}\right\}}$$

Q) Find the fourier cosine transform of

$$f(x) = \begin{cases} x^2, & 0<x<1 \\ 0, & x>1 \end{cases}$$

Soln:

Fourier cosine transform of $f(x)$ is,

$$f_c^{\wedge}(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\omega x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^1 x^2.\cos\omega x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left[x^2.\frac{\sin\omega x}{\omega}\right]_0^1 - \int_0^1 2x\,\frac{\sin\omega x}{\omega}\, dx\right.$$

$$\Rightarrow f_c^\wedge(w) = \sqrt{\frac{2}{\pi}}\left\{\left(\frac{\sin w}{w}\right) - \frac{2}{w}\int_0^1 x \sin wx \, dx\right.$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left(\frac{\sin w}{w}\right) - \frac{2}{w}\left[\left[x \cdot \frac{-\cos wx}{w}\right]_0^1 - \int_0^1 \frac{-\cos wx}{w}\, dx\right]\right.$$

$$= \sqrt{\frac{2}{\pi}}\left\{\left(\frac{\sin w}{w}\right) - \frac{2}{w}\left[-\left(\frac{\cos w}{w}\right) + \left(\frac{\sin w}{w^2}\right)\right]_0^1\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{\frac{\sin w}{w} + \frac{2\cos w}{w^2} - \frac{2\sin w}{w^3}\right\}$$

$$\boxed{f_c^\wedge(w) = \sqrt{\frac{2}{\pi}}\left\{\frac{w^2 \sin w + 2w\cos w - 2\sin w}{w^3}\right\}}$$

Q) find the fourier cosine transform of $f(x) = \sin x$, $0 < x < \pi$.

Soln:-

fourier cosine transform,

$$f_c^\wedge(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x) \cdot \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\pi \sin x \cos wx \, dx = \sqrt{\frac{2}{\pi}}\int_0^\pi \left[\frac{\sin(x+wx) + \sin(x-wx)}{2}\right] dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{2}\int_0^\pi \left[\sin(1+w)x + \sin(1-w)x\right] dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{2}\left\{\int_0^\pi \sin(1+w)x + \int_0^\pi \sin(1-w)x \, dx\right\}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2}\left\{\left[\frac{-\cos(1+w)x}{1+w}\right]_0^\pi + \left[\frac{-\cos(1-w)x}{1-w}\right]_0^\pi\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{-\left(\frac{\cos(\pi+\pi w) - 1}{1+w}\right) - \left(\frac{\cos(\pi-\pi w) - 1}{1-w}\right)\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{-\left(\frac{-\cos\pi w - 1}{1+w}\right) - \left(\frac{-\cos\pi w - 1}{1-w}\right)\right\}$$

$\cos(\pi+\theta) = -\cos\theta$

$\cos(\pi-\theta) = -\cos\theta$

$$\Rightarrow f_c^\wedge(\omega) = \frac{1}{\sqrt{2\pi}}\left\{\frac{(1+\cos\pi\omega)}{1+\omega} + \frac{(1+\cos\pi\omega)}{1-\omega}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{(1-\omega)(1+\cos\pi\omega) + (1+\omega)(1+\cos\pi\omega)}{(1+\omega)(1-\omega)}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{(1+\cos\pi\omega - \omega - \omega\cos\pi\omega) + (1+\cos\pi\omega + \omega + \omega\cos\pi\omega)}{1-\omega^2}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{2+2\cos\pi\omega}{1-\omega^2}\right\} = \frac{2}{\sqrt{2\pi}}\left\{\frac{1+\cos\pi\omega}{1-\omega^2}\right\}$$

$$\Rightarrow \boxed{f_c^\wedge(\omega) = \sqrt{\frac{2}{\pi}}\left[\frac{1+\cos\pi\omega}{1-\omega^2}\right]}$$

Q) find the fourier cosine transform of

$$f(x) = \begin{cases} \cos x, & 0<x<1 \\ 0 & x\geq 1 \end{cases}$$

soln:

fourier cosine transform of $f(x)$,

$$f_c^\wedge(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left\{\int_0^1 \cos x\cos\omega x\, dx\right\} = \sqrt{\frac{2}{\pi}}\int_0^1\left[\frac{\cos(x+\omega x)+\cos(x-\omega x)}{2}\right]dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{2}\left\{\int_0^1 \cos(1+\omega)x + \int_0^1 \cos(1-\omega)x \, dx\right.$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{2}\left\{\left[\frac{\sin(1+\omega)x}{1+\omega}\right]_0^1 + \left[\frac{\sin(1-\omega)x}{1-\omega}\right]_0^1\right\}$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{2}\left\{\frac{\sin(1+\omega)}{1+\omega} + \frac{\sin(1-\omega)}{1-\omega}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{(1-\omega)\sin(1+\omega) + (1+\omega)\sin(1-\omega)}{1-\omega^2}\right\}$$

Q) Find the fourier sine transform of $f(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>0 \end{cases}$

Soln:

fourier sine transform of $f(x)$,

$$f_s^1(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \sin \omega x \, dx + \int_1^2 (2-x) \sin \omega x \, dx \right]$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \left\{ \left[\left[x \cdot \frac{-\cos \omega x}{\omega} \right]_0^1 - \int_0^1 \frac{-\cos \omega x}{\omega} \right] + \left[(2-x) \cdot \frac{-\cos \omega x}{\omega} \right]_1^2 - \int_1^2 -1 \cdot \frac{-\cos \omega x}{\omega} dx \right.$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left[-\frac{\cos \omega}{\omega} + \left[\frac{\sin \omega x}{\omega^2} \right]_0^1 \right] + \left[-\left(0 - \frac{\cos \omega}{\omega}\right) - \left[\frac{\sin \omega x}{\omega^2} \right]_1^2 dx \right. \right.$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left[\cancel{-\frac{\cos \omega}{\omega}} + \frac{\sin \omega}{\omega^2} \right] + \left[\cancel{\frac{\cos \omega}{\omega}} - \frac{(\sin 2\omega - \sin \omega)}{\omega^2} \right] \right.$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{\sin \omega}{\omega^2} - \frac{(\sin 2\omega - \sin \omega)}{\omega^2} \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{\sin \omega}{\omega^2} - \frac{\sin 2\omega}{\omega^2} + \frac{\sin \omega}{\omega^2} \right\}$$

$$\Rightarrow \boxed{f_s^1(\omega) = \sqrt{\frac{2}{\pi}} \left\{ \frac{2\sin \omega - \sin 2\omega}{\omega^2} \right\}}$$

MODULE - IV

LAPLACE TRANSFORMS

Definition:- Let $f(t)$ be a function of t. $t>0$ then the Laplace transform of $f(t)$ denoted by $L\{f(t)\}$ and is defined by $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$, s is real number.

Q) Find Laplace transform of i) $f(t)=0$

ii) $f(t)=1$

Solution:-

Laplace transform of $f(t)$,

$$L\{f(t)\} = L\{0\} = \int_0^\infty e^{-st}.\, f(t)\,dt$$

$$= \int_0^\infty e^{-st}.\, 0\,dt$$

$$= 0$$

$$\boxed{\therefore\ L\{0\} = 0}$$

ii) $f(t) = 1$

$$L\{f(t)\} = L\{1\} = \int_0^\infty e^{-st}.\, f(t)\,dt$$

$$= \int_0^\infty e^{-st}.\, 1\,dt$$

$$= \left.\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{-1}{s}(0-1)$$

$$= \frac{1}{s}$$

$$\boxed{\therefore\ L\{1\} = \frac{1}{s}}$$

Q) Find the Laplace transform of $f(t) = t^n$.

solution:-

Laplace transform of $f(t)$,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$\therefore L\{t^n\} = \int_0^\infty e^{-st} \cdot t^n\, dt$$

$$= \int_0^\infty t^n \cdot e^{-st}\, dt \quad \text{Integration by parts,}$$

$$= \left[\left(t^n \cdot \frac{e^{-st}}{-s}\right)_0^\infty - \int_0^\infty n t^{n-1} \cdot \frac{e^{-st}}{-s}\, dt\right.$$

$$= 0 + \frac{n}{s}\int_0^\infty t^{n-1} \cdot e^{-st}\, dt$$

ie $L\{t^n\} = \frac{n}{s} L\{t^{n-1}\}$

Similarly, $L\{t^{n-1}\} = \frac{n-1}{s} L\{t^{n-2}\}$

$L\{t^{n-2}\} = \frac{n-2}{s} L\{t^{n-3}\}$

$\vdots$

$\therefore$ $L\{t^2\} = \frac{2}{s} \cdot L\{t^1\}$

$L\{t^1\} = \frac{1}{s} \cdot L\{t^{1-1}\}$

ie $= \frac{1}{s} \cdot L\{1\}$

ie $L\{t^1\} = \frac{1}{s} \cdot \frac{1}{s}$ $\qquad \therefore L\{1\} = \frac{1}{s}$

$$\therefore L\{t^n\} = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots \frac{2}{s} \cdot \frac{1}{s} \cdot \frac{1}{s}$$

$$= \left[\frac{n \cdot n-1 \cdot n-2 \cdots 2 \cdot 1}{s \cdot s \cdot s \cdots n\,\text{times}}\right] \cdot \frac{1}{s}$$

$$= \frac{n!}{s^n} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}$$

$$\boxed{\therefore L\{t^n\} = \frac{n!}{s^{n+1}}}$$

Q) Find $L\{t^3\}$

solution:-

we've $L\{t^n\} = \frac{n!}{s^{n+1}}$

$\therefore L\{t^3\} = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$

Q) Find $L\{t\}$

solution:-

We've $L\{t^n\} = \frac{n!}{s^{n+1}}$

$\therefore L\{t\} = \frac{1}{s^{1+1}} = \frac{1}{s^2}$

Q) Find $L\{e^{at}\}$ and $L\{e^{-at}\}$

solution:-

$$L\{e^{at}\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$= \int_0^\infty e^{-st} \cdot e^{at}\, dt$$

$$= \int_0^\infty e^{-t(s-a)}\, dt$$

$$= \left. \frac{e^{-t(s-a)}}{-(s-a)} \right]_0^\infty$$

$$= \frac{-1}{s-a}\left[0-1\right] = \frac{1}{s-a}$$

$$\therefore \boxed{L\{e^{at}\} = \frac{1}{s-a}}$$

$$L\{e^{-at}\} = \int_0^\infty e^{-st} \cdot e^{-at}\, dt$$

$$= \int_0^\infty e^{-t(s+a)}\, dt$$

$$L\{e^{-at}\} = \left. \frac{e^{-t(s+a)}}{-(s+a)} \right]_0^\infty$$

$$= \frac{1}{-(s+a)}\left[0-1\right] = \frac{1}{s+a}$$

$$\boxed{L\{e^{-at}\} = \frac{1}{s+a}}$$

Q) Find $L\{e^{3t}\}$ and $L\{e^{-2t}\}$

Soln: we've $L\{e^{at}\} = \frac{1}{s-a}$

$\therefore L\{e^{3t}\} = \frac{1}{s-3}$, $a = 3$

$\therefore L\{e^{-2t}\} = \frac{1}{s+2}$, $a = -2$

Q) Find $L\{\sinh at\}$ and $L\{\cosh at\}$

Soln:

Result :- $\sinh at = \frac{e^{at} - e^{-at}}{2}$

$\cosh at = \frac{e^{at} + e^{-at}}{2}$

Laplace transform of $f(t)$,

$$L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt.$$

$$L\{\sinh at\} = L\left[\frac{e^{at} - e^{-at}}{2}\right]$$

$$= \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-(s-a)}{s^2-a^2}\right]$$

$$= \frac{1}{2}\,\frac{2a}{s^2-a^2} = \frac{a}{s^2-a^2}$$

$$\boxed{L\{\sinh at\} = \frac{a}{s^2-a^2}}$$

$$L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\left[L\{e^{at}\} + L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{(s+a)+(s-a)}{s^2-a^2}\right] = \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$$

$$\boxed{\therefore L\{\cosh at\} = \frac{s}{s^2-a^2}}$$

Q) Find $L\{f(t)\}$ if $f(t) = \begin{cases} e^{-t}, & 0<t<4 \\ 0, & t>4 \end{cases}$

soln:-

Laplace transform of $f(t) = \int_0^\infty e^{-st} f(t)\, dt$.

$$L\{f(t)\} = \int_0^4 e^{-st} \cdot e^{-t}\, dt$$

$$= \int_0^4 e^{-t(s+1)}\, dt$$

$$= \left.\frac{e^{-t(s+1)}}{-(s+1)}\right]_0^4 = \frac{e^{-4(s+1)} - 1}{-(s+1)}$$

$$= \frac{1-e^{-4(s+1)}}{s+1}$$

Q) Find the Laplace transform of $f(t)$ if $f(t) = \begin{cases} (t-1)^2, & t>1 \\ 0, & t<1 \end{cases}$

soln:-

$$L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$= \int_1^\infty e^{-st} (t-1)^2\, dt \, .$$

$$= \int_1^\infty (t-1)^2 \cdot e^{-st}\, dt$$

Integration by parts,

$$L\{f(t)\} = \left[\left((t-1)^2 \cdot \frac{e^{-st}}{-s}\right)_1^\infty - \int_1^\infty 2(t-1)\cdot\frac{e^{-st}}{-s}\, dt\right]$$

$$= 0 + \frac{2}{s}\int_1^\infty (t-1)\, e^{-st}\, dt$$

$$= \frac{2}{s}\left[\left((t-1)\cdot\frac{e^{-st}}{-s}\right)_1^\infty - \int_1^\infty \frac{e^{-st}}{-s}\, dt\right]$$

$$= \frac{2}{s}\left[0 + \frac{1}{s}\int_1^\infty e^{-st}\, dt\right]$$

$$= \frac{2}{s}\left[\frac{1}{s}\cdot\left(\frac{e^{-st}}{-s}\right)_1^\infty\right]$$

$$= \frac{2}{s}\left[\frac{-1}{s^2}\left(0 - e^{-s}\right)\right]$$

$$L\{f(t)\} = \underline{\underline{\frac{2}{s^3} e^{-s}}}$$

Q) Find the Laplace transform of $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

<u>solution</u>:-

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\pi e^{-st} \cdot \sin t\, dt.$$

This can be solved by using the result

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left(a \sin bx - b\cos bx\right)$$

$$\therefore L\{f(t)\} = \left[\frac{e^{-st}}{(-s)^2+1^2}\left(-s \sin 1\cdot t - 1\cdot\cos 1\cdot t\right)\right]_0^\pi \qquad \begin{matrix} a = -s \\ b = 1 \end{matrix}$$

$$= \left[\frac{e^{-s\pi}}{s^2+1}\left(-s\sin\pi - \cos\pi\right)\right] - \frac{1}{s^2+1}\left(0-1\right)$$

$\therefore L\{f(t)\} = \frac{1}{s^2+1}\left[e^{-s\pi}(0 - {-1}) + 1\right]$

$\boxed{\begin{array}{l}\sin\pi = 0\\ \cos\pi = -1\end{array}}$

$\therefore L\{\sin t\} = \frac{1}{s^2+1}\left(e^{-s\pi}+1\right)$

FIRST SHIFTING THEOREM - SSHIFTING

If $f(t)$ has the Laplace transform

$F(s) = L\{f(t)\}$,

then $\boxed{F(s-a) = L\{e^{at} \cdot f(t)\} = \int_0^\infty e^{-st} \cdot \left[e^{at} \cdot f(t)\right] dt.}$

and $\boxed{F(s+a) = L\{e^{-at} \cdot f(t)\} = \int_0^\infty e^{-at} \cdot e^{-st} \cdot f(t)\, dt.}$

Results :-

1. $L\{e^{at}\} = \frac{1}{s-a}$
2. $L\{e^{at} \cdot t^n\} = \frac{n!}{(s-a)^{n+1}}$ since $L\{t^n\} = \frac{n!}{s^{n+1}}$
3. $L\{e^{at} \cdot \sin bt\} = \frac{b}{(s-a)^2+b^2}$ $\quad L\{\sin bt\} = \frac{b}{s^2+b^2}$
4. $L\{e^{at} \cdot \cos bt\} = \frac{s-a}{(s-a)^2+b^2}$ $\quad L\{\cos bt\} = \frac{s}{s^2+b^2}$
5. $L\{e^{at} \cdot \sinh bt\} = \frac{b}{(s-a)^2-b^2}$ $\quad L\{\sinh bt\} = \frac{b}{s^2-b^2}$
6. $L\{e^{at} \cdot \cosh bt\} = \frac{s-a}{(s-a)^2-b^2}$ $\quad L\{\cosh at\} = \frac{s}{s^2-b^2}$

* INVERSE LAPLACE TRANSFORMS

1. $L^{-1}\left\{\frac{1}{s}\right\} = 1$

2. $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$

3. $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

4. $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

5. $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

6. $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$

7. $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

8. $L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b}e^{at}\sin bt$

9. $L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$

10. $L^{-1}\left\{\frac{1}{(s-a)^2-b^2}\right\} = \frac{1}{b}e^{at}\sinh bt$

11. $L^{-1}\left\{\frac{(s-a)}{(s-a)^2-b^2}\right\} = e^{at}\cosh at$

Q) Find $L^{-1}\left\{\frac{1}{s-3}\right\}$.

Soln:-

$$L\{e^{at}\} = \frac{1}{s-a}$$

$$\therefore L\{e^{3t}\} = \frac{1}{s-3}$$

$\therefore$ Inverse Laplace transform

$$L^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$$

Q) Find $L^{-1}\left\{\frac{1}{s^2-25}\right\}$

Soln:

$$L^{-1}\left\{\frac{1}{s^2-25}\right\} = L^{-1}\left\{\frac{1}{s^2-5^2}\right\}$$

$$= \frac{1}{5}\sinh 5t$$

$\because$ $L^{-1}\left\{\frac{1}{s^2-b^2}\right\} = \frac{1}{b}\sinh bt$

here $b=5$

Q) Find $L^{-1}\left\{\frac{1}{(s-2)^2+1}\right\}$

Soln:-

$$L^{-1}\left\{\frac{1}{(s-2)^2+1}\right\} = L^{-1}\left\{\frac{1}{(s-2)^2+1^2}\right\}$$

$$= \frac{1}{1}\cdot e^{2t}\sin 1\cdot t$$

$$= e^{2t}\sin t$$

$\because$ $L^{-1}\frac{1}{(s-a)^2+b^2} = \frac{1}{b}e^{at}\sin bt$

here $a=2$, $b=1$

Q) Find $L^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\}$

Soln:-

$$L^{-1}\left\{\frac{s-1}{(s-1)^2+2^2}\right\} = e^{1\cdot t}\cos 2t = e^{t}\cos 2t$$

here $\because$ $L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$

Q) $L^{-1}\left\{\frac{2s}{s^2-16}\right\}$

Soln:

$$L^{-1}\left\{\frac{2s}{s^2-4^2}\right\} = 2L^{-1}\left\{\frac{s}{s^2-4^2}\right\}$$

$\therefore L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

$$= 2\cosh 4t$$

Q) Find $L^{-1}\left\{\frac{2s^2-4s+5}{s^3}\right\}$

Soln:

$$L^{-1}\left\{\frac{2s^2-4s+5}{s^3}\right\} = L^{-1}\left\{\frac{2s^2}{s^3}\right\} - L^{-1}\left\{\frac{4s}{s^3}\right\} + L^{-1}\left\{\frac{5}{s^3}\right\}$$

$$= 2L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{4}{s^2}\right\} + L^{-1}\left\{\frac{5}{s^3}\right\}$$

$$= 2.L^{-1}\left\{\frac{1}{s}\right\} - 4L^{-1}\left\{\frac{1}{s^2}\right\} + 5L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= 2\times(1) \quad - 4.(t) + 5\cdot\frac{1}{2}(t^2) =$$

$$= 2 - 4t + \frac{5}{2}t^2$$

$L\{1\} = \frac{1}{s}$

$L\{t\} = \frac{1}{s}$

$L\{t^2\} = \frac{2!}{s^3}$

Q) Find $L^{-1}\left\{\frac{s-3}{s^2+4s+13}\right\}$

soln:- $L^{-1}\left\{\frac{s-3}{(s+2)^2+9}\right\} = L^{-1}\left\{\frac{s+2-5}{(s+2)^2+9}\right\}$

$$= L^{-1}\left\{\frac{(s+2)}{(s+2)^2+3^2} - \frac{5}{(s+2)^2+3^2}\right\}$$

$$= L^{-1}\left\{\frac{(s-^-2)}{(s-^-2)^2+3^2} - \frac{5}{(s-^-2)^2+3^2}\right\}$$

$$= L^{-1}\left\{\frac{(s-^-2)}{(s-^-2)^2+3^2}\right\} - 5L^{-1}\left\{\frac{1}{(s-^-2)^2+3^2}\right\}$$

$$= L^{-1}\left\{ e^{-2t} \cdot \cos 3t - 5 \cdot \frac{1}{3} \cdot e^{-2t} \cdot \sin 3t \right.$$

$$= e^{-2t} \cos 3t - \frac{5}{3} e^{-2t} \cdot \sin 3t \qquad \left[\because L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = e^{at} \sin bt \right.$$

$a = -2, \ b = 3$

Q) Find $L^{-1}\left\{\frac{s}{(s+2)^2+1}\right\}$

Soln:

$$L^{-1}\left\{\frac{s}{(s+2)^2+1}\right\} = L^{-1}\left\{\frac{s+2-2}{(s+2)^2+1^2}\right\}$$

$$= L^{-1}\left\{\frac{s+2}{(s+2)^2+1^2}\right\} - 2L^{-1}\left\{\frac{1}{(s+2)^2+1^2}\right\}$$

$$= e^{-2t} \cos 1t - 2 e^{-2t} \sin 1 \cdot t$$

$$= e^{-2t} \cos t - 2 e^{-2t} \sin t \qquad \left[\text{here } a = -2, \ b = 1 \right.$$

Q) Find $L^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\}$

Soln:-

$$\frac{1}{(s-a)(s-b)} = \frac{A}{s-a} + \frac{B}{s-b} \quad \text{using partial fractions}$$

$$1 = A(s-b) + B(s-a)$$

Put $s = a \Rightarrow 1 = A(a-b)$

$$\Rightarrow A = \frac{1}{a-b}$$

Put $s = b \Rightarrow 1 = B(b-a)$

$$\Rightarrow B = \frac{1}{b-a}$$

$$\therefore L^{-1}\left\{\frac{1}{s-a} + \frac{1}{s-b}\right\} = L^{-1}\left\{\frac{\left(\frac{1}{a-b}\right)}{s-a} + \frac{\left(\frac{1}{b-a}\right)}{s-b}\right\} = \frac{1}{a-b} \cdot e^{at} + \frac{1}{b-a} \cdot e^{bt}$$

Q) P.T $L\{f'(t)\} = sL\{f(t)\} - f(0)$

Soln:-

$$L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\,dt$$

$$L\{f'(t)\} = \int_0^\infty e^{-st} \cdot f'(t)\,dt$$

$$= \left\{ \left(e^{-st} \cdot f(t)\right)_0^\infty - \int_0^\infty -s \cdot e^{-st} f(t)\,dt \right.$$

$$= 0 - f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

ie $L\{f'(t)\} = -f(0) + sL(f\{t\})$

$$\boxed{\text{ie} \quad L\{f'(t)\} = sL\{f(t)\} - f(0)} \quad \text{— (1)}$$

Q) P.T $L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$

Soln:

$$L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\,dt$$

$$L\{f''(t)\} = \int_0^\infty e^{-st} \cdot f''(t)\,dt$$

$$= \left\{ \left(e^{-st} \cdot f'(t)\right)_0^\infty - \int_0^\infty -s \cdot e^{-st} \cdot f'(t)\,dt \right.$$

$$L\{f''(t)\} = -f'(0) + s\int_0^\infty e^{-st} f'(t)$$

$$= -f'(0) + s\left[L\{f'(t)\}\right]$$

ie $L\{f''(t)\} = -f'(0) + s\left[sL\{f(t)\} - f(0)\right]$ using result (1).

$$L\{f''(t)\} = -f'(0) + s^2 L\{f(t)\} - sf(0)$$

$$\boxed{\therefore \quad L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)}$$

Q) Find $L^{-1}\left\{\frac{9}{s^2+3s}\right\}$

Soln:-

$L^{-1}\left\{\frac{9}{s(s+3)}\right\}$ =

Using partial fractions,

$\frac{9}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$

$\Rightarrow 9 = A(s+3) + Bs$

Put $s=0 \Rightarrow 9 = A(0+3)$

$\Rightarrow A = \frac{9}{3} = 3$

Put $s=-3 \Rightarrow 9 = -3B$

$\Rightarrow B = -3$

$\therefore L^{-1}\left\{\frac{9}{s(s+3)}\right\} = L^{-1}\left\{\frac{3}{s} + \frac{-3}{s+3}\right\}$

$= 3L^{-1}\left\{\frac{1}{s}\right\} + -3L^{-1}\left\{\frac{1}{s+3}\right\}$

$= 3\cdot(1) - 3(e^{-3t})$

$= 3 - 3e^{-3t}$

Q) Find $L^{-1}\left\{\frac{3s}{s^2+2s-8}\right\}$

Soln:

$\frac{3s}{s^2+2s-8} = \frac{3s}{(s+4)(s-2)} = \frac{A}{s+4} + \frac{B}{s-2}$

$3s = A(s-2) + B(s+4)$

Put $s=2 \Rightarrow 6 = B(2+4)$

$\Rightarrow B = 1$

Put $s = -4 \Rightarrow -12 = -6A$

$\Rightarrow A = 2$

$$\therefore L^{-1}\left\{\frac{3s}{s^2+2s-8}\right\} = L^{-1}\left\{\frac{3s}{(s+4)(s-2)}\right\}$$

$$= L^{-1}\left\{\frac{2}{s+4} + \frac{1}{s-2}\right\}$$

$$= L^{-1}\left\{\frac{2}{s+4}\right\} + L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 2e^{-4t} + e^{2t}$$

Q) Find $L^{-1}\left\{\frac{s-3}{s^2-1}\right\}$

Soln:-

$$\frac{s-3}{s^2-1} = \frac{s-3}{(s-1)(s+1)} = \frac{s-3}{(s+1)(s+1)}$$

$$\frac{s-3}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1}$$

$$s-3 = A(s-1) + B(s+1)$$

put $s=1 \Rightarrow -2 = 2B \Rightarrow B = -1$

put $s=-1 \Rightarrow -4 = -2A \Rightarrow A = 2$

$$\therefore L^{-1}\left\{\frac{s-3}{(s+1)(s-1)}\right\} = L^{-1}\left\{\frac{2}{s+1} + \frac{-1}{s-1}\right\}$$

$$= L^{-1}\left\{\frac{2}{s+1}\right\} - L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= 2e^{-t} - e^{t}$$

Q) Find $L^{-1}\left\{\frac{1}{s(s-1)(s-2)}\right\}$

Soln:- $\frac{1}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{(s-2)}$

$$1 = A(S-1)(S+2) + B\,S(S+2) + C(S)(S-1)$$

put $S=0 \Rightarrow \quad -2A = 1 \Rightarrow A = -\frac{1}{2}$

put $S=1 \Rightarrow \quad 3B = 1 \Rightarrow B = \frac{1}{3}$

put $S=-2 \Rightarrow \quad 6C = 1 \Rightarrow C = \frac{1}{6}$

$$\therefore L^{-1}\left\{\frac{1}{S(S-1)(S-2)}\right\} = L^{-1}\left\{\frac{-1/2}{S} + \frac{1/3}{S-1} + \frac{1/6}{S+2}\right\}$$

$$= -\frac{1}{2}L^{-1}\left\{\frac{1}{S}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{S-1}\right\} + \frac{1}{6}L^{-1}\left\{\frac{1}{S+2}\right\}$$

$$= -\frac{1}{2}(1) + \frac{1}{3}(e^{t}) + \frac{1}{6}e^{-2t}$$

$$\text{ie } L^{-1}\left\{\frac{1}{S(S-1)(S-2)}\right\} = \frac{-1}{2} + \frac{e^{t}}{3} + \frac{e^{-2t}}{6}$$

Q) Find $L^{-1}\left\{\frac{5S^2+3S-16}{(S-1)(S-2)(S+3)}\right\}$

soln:-

$$\frac{5S^2+3S-16}{(S-1)(S-2)(S+3)} = \frac{A}{S-1} + \frac{B}{S-2} + \frac{C}{S+3}$$

$$5S^2+3S-16 = A(S-2)(S+3) + B(S-1)(S+3) + C(S-1)(S-2)$$

put $S=1 \Rightarrow -8 = -4A \Rightarrow A = +2$

put $S=2 \Rightarrow 10 = 5B \Rightarrow B = 2$

put $S=-3 \Rightarrow 20 = 20C \Rightarrow C = 1$

$$\therefore L^{-1}\left\{\frac{5S^2+3S-16}{(S-1)(S-2)(S+3)}\right\} = L^{-1}\left\{\frac{+2}{S-1} + \frac{2}{S-2} + \frac{1}{S+3}\right\}$$

$$= +2L^{-1}\left\{\frac{1}{S-1}\right\} + 2L^{-1}\left\{\frac{1}{S-2}\right\} + L^{-1}\left\{\frac{1}{S+3}\right\}$$

$$= +2e^{t} + 2e^{2t} + e^{-3t}$$

Q) Find $L^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\}$

solution:-

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3}$$

$$5s^2-15s-11 = A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1)$$

put $s=2 \Rightarrow -21 = 3D \Rightarrow D = -7$

put $s=-1 \Rightarrow 9 = -27A \Rightarrow A = -\frac{1}{3}$

equating coefficient of s^3, $0 = A+B \Rightarrow A = -B \Rightarrow B = \frac{1}{3}$

put $s=0 \Rightarrow$
$$-11 = -8A+4B-2C+D$$
$$-11 = -8\left(-\frac{1}{3}\right)+4B-2C-7$$
$$-11 = \frac{8}{3}+4B-2C-7$$
$$-11 = \frac{8}{3}+4\left(\frac{1}{3}\right)-2C-7 \Rightarrow C = 4$$

$$\therefore L^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\} = L^{-1}\left\{\frac{-1/3}{s+1} + \frac{1/3}{s-2} + \frac{4}{(s-2)^2} + \frac{-7}{(s-2)^3}\right\}$$

$$= L^{-1}\left\{\frac{-1/3}{s+1}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{s-2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2}\right\} - 7L^{-1}\left\{\frac{1}{(s-2)^3}\right\}$$

$$= \frac{-1}{3}e^{t} + \frac{1}{3}e^{2t} + 4t\cdot e^{2t} - \frac{7}{2}t^2e^{2t}$$

Result:-

$$L^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$L^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$$

similarly $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$ and $L^{-1}\left\{\frac{1}{(s-a)^2}\right\} = te^{at}$

also $L^{-1}\left\{\frac{1}{(s-a)^3}\right\} = \frac{t^2}{2!}e^{at}$

INITIAL VALUE PROBLEM

$$\boxed{\begin{aligned} L\{y''\} &= s^2 L\{y\} - s y\{0\} - y'\{0\} \\ L\{y'\} &= s L\{y\} - y\{0\}. \end{aligned}}$$

Q) Solve the initial value problem

$y'' - 2y' + 10y = 0$, $y(0) = 3$, $y'(0) = 3$

Solution:-

Taking Laplace transform on both sides,

$$L\{y'' - 2y' + 10y\} = L\{0\}$$

$$L\{y''\} - 2L\{y'\} + 10L\{y\} = L\{0\}$$

$$s^2 L\{y\} - s\,y(0) - y'(0) - 2\left[sL\{y\} - y(0)\right] + 10L\{y\} = 0$$

$$(s^2 - 2s + 10)\,L\{y\} - s(3) - (3) + 2\cdot(3) = 0$$

$$(s^2 - 2s + 10)\,L\{y\} = 3s - 3$$

$$\therefore \quad L\{y\} = \frac{3s-3}{s^2 - 2s + 10}$$

$$\Rightarrow y = L^{-1}\left\{\frac{3s-3}{s^2-2s+10}\right\} = L^{-1}\left\{\frac{3(s-1)}{s^2-2s+1+9}\right\}$$

$$\Rightarrow \quad = 3L^{-1}\left\{\frac{(s-1)}{(s-1)^2+3^2}\right\}$$

$$L^{-1}\left\{\frac{3(s-1)}{(s-1)^2+3^2}\right\} = 3e^{t}\cos 3t$$

$$\boxed{\because L^{-1}\left\{\frac{(s-a)}{(s-a)^2+b^2}\right\} = e^{at}\cos bt.}$$

Q) Solve $y''+2y'+2y=0 \quad y(0)=0, \; y'(0)=1$

soln:-

Taking Laplace on both sides,

$L\{y''+2y'+2y\} = L\{0\}$

$L\{y''\}+2L\{y'\}+2L\{y\} = L\{0\}$

$s^2L\{y\}-sy(0)-y'(0)+2[sL\{y\}-y(0)]+2L\{y\}=0$

$(s^2+2s+2)L\{y\}-s\cdot y(0)-y'(0)-2y(0)=0$

$(s^2+2s+2)L\{y\}-s\cdot(0)-(1)-2(0)=0$

$L\{y\}(s^2+2s+2) = 1$

$L\{y\} = \frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1^2}$

$\therefore y = L^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\}$

$= e^{-t}\sin t$

$$\boxed{\begin{array}{l}\text{since } L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = e^{at}\sin bt. \\ \text{here } a=-1,\; b=1\end{array}}$$

Q) Solve the initial value problem

$y''+2y'-3y=3 \quad , y(0)=4 \quad , y'(0)=-7$

soln:-

Taking Laplace transform on both sides,

$L\{y''\}+2L\{y'\}-3L\{y\}=L\{3\}$

$s^2L\{y\}-s\{y(0)\}-y'(0)+2[sL\{y\}-y(0)]-3L\{y\}=3L\{1\}$

$L\{y\}[s^2+2s-3] = (3\cdot\frac{1}{s})+(4s+1)$

$L\{y\}[s^2+2s-3] = \frac{3}{s}+(4s+1)$

$$\therefore\ L\{y\} = \frac{\frac{3}{s}}{s^2+2s-3} + \frac{4s+1}{s^2+2s-3}$$

$$\text{ie } L\{y\} = \frac{3}{s\,(s-1)(s-3)} + \frac{4s+1}{(s-1)(s-3)}$$

$$\therefore\ y = L^{-1}\left\{\frac{3}{s(s-1)(s+3)}\right\} + L^{-1}\left\{\frac{4s+1}{(s-1)(s+3)}\right\}$$

Using partial fractions,

$$\frac{3}{s(s-1)(s+3)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+3}$$

$$\Rightarrow\ 3 = A(s-1)(s+3) + Bs(s+3) + C(s)(s-1)$$

put $s=1 \Rightarrow 3 = 4B \Rightarrow B = 3/4$

put $s=0 \Rightarrow -3A = 3 \Rightarrow A = -1$

put $s=-3 \Rightarrow 3 = 12C \Rightarrow C = 1/4$

$$\therefore L^{-1}\left\{\frac{3}{s(s-1)(s+3)}\right\} = L^{-1}\left\{\frac{-1}{s} + \frac{3/4}{s-1} + \frac{1/4}{s+3}\right\}$$

$$= L^{-1}\left\{\frac{-1}{s}\right\} + L^{-1}\left\{\frac{3/4}{s-1}\right\} + L^{-1}\left\{\frac{1/4}{s+3}\right\}$$

$$= -L^{-1}\left\{\frac{1}{s}\right\} + \frac{3}{4}L^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{s+3}\right\}$$

also $L^{-1}\left\{\dfrac{4s+1}{(s-1)(s+3)}\right\}$

Using partial fractions,

$$\frac{4s+1}{(s-1)(s+3)} = \frac{A}{(s-1)} + \frac{B}{(s+3)}$$

$$\Rightarrow\ 4s+1 = A(s+3) + B(s-1)$$

put $s=1 \Rightarrow 5 = 4A \Rightarrow A = \frac{5}{4}$

put $s=-3 \Rightarrow -11 = -4B \Rightarrow B = \frac{11}{4}$

$$\therefore L^{-1}\left\{\frac{4s+1}{(s-1)(s+3)}\right\} = L^{-1}\left\{\frac{5/4}{s-1} + \frac{11/4}{s+3}\right\}$$

$$= \frac{5}{4} L^{-1}\left\{\frac{1}{s-1}\right\} + \frac{11}{4} L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$\therefore L^{-1}\left\{\frac{3}{s(s-1)(s+3)}\right\} + L^{-1}\left\{\frac{4s+1}{(s-1)(s+3)}\right\}$$

$$= L^{-1}\left\{\frac{-1}{s} + \frac{3}{4}\cdot\frac{1}{s-1} + \frac{1}{4}\frac{1}{(s+3)} + \frac{5}{4}\frac{1}{s-1} + \frac{11}{4}\frac{1}{s+3}\right\}$$

$$= L^{-1}\left\{\frac{-1}{s} + \frac{2}{s-1} + \frac{3}{s+3}\right\}$$

$$= -L^{-1}\left\{\frac{1}{s}\right\} + 2L^{-1}\left\{\frac{1}{s-1}\right\} + 3L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= -1 + 2e^{t} + 3e^{-3t}$$

Q) Solve $y'' - y = t$, $y(0)=1$, $y'(0)=1$

Soln:-

Taking Laplace on both sides,

$$L\{y'' - y\} = L\{t\}$$

$$L\{y''\} - L\{y\} = L\{t\}$$

$$s^2 L\{y\} - s\,y(0) - y'(0) - L\{y\} = L\{t\}$$

$$L\{y\}\left[s^2 - 1\right] = \frac{1}{s^2} + (s+1)$$

$$\therefore L\{y\} = \frac{1}{s^2(s^2-1)} + \frac{s+1}{(s^2-1)}$$

$$\text{ie } y = L^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\} + L^{-1}\left\{\frac{s}{(s^2-1)}\right\}.$$

$$\text{Now } \frac{1}{s^2(s^2-1)} = \frac{1}{s^2(s+1)(s-1)}$$

using partial fractions,

$$\frac{1}{s^2(s+1)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s+1)} + \frac{D}{(s-1)}$$

$$\Rightarrow 1 = As(s+1)(s-1) + B(s+1)(s-1) + C(s^2)(s-1) + D s^2 (s+1)$$

put $s=1 \Rightarrow 1 = 2D \Rightarrow D = 1/2$

put $s=-1 \Rightarrow 1 = -2C \Rightarrow C = -1/2$

put $s=0 \Rightarrow 1 = -B \Rightarrow B = -1$

equating coeff of s^3,

$$0 = A + C + D$$

$$\Rightarrow 0 = A + \frac{-1}{2} + \frac{1}{2} \Rightarrow A = 0$$

$$\therefore L^{-1}\left\{\frac{1}{s^2(s+1)(s-1)}\right\} = L^{-1}\left\{\frac{0}{s} + \frac{-1}{s^2} + \frac{-1/2}{s+1} + \frac{1/2}{s-1}\right\}$$

$$= L^{-1}\{0\} + -L^{-1}\left\{\frac{1}{s^2}\right\} + \frac{-1}{2}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$L^{-1}\left\{\frac{1}{s^2(s+1)(s-1)}\right\} = 0 + -t + \frac{-1}{2}\cdot e^{-t} + \frac{1}{2}\cdot e^{t}$$

also $L^{-1}\left\{\frac{1}{s-1}\right\} = e^t$

$$\therefore L^{-1}\left\{\frac{1}{s^2(s+1)(s-1)} + \frac{1}{s-1}\right\} = \left(-t + \frac{-1}{2}e^{-t} + \frac{1}{2}e^{t}\right) + \left(e^t\right)$$

CONVOLOTION THEOREM

If $L^{-1}\{f(s)\} = f(t)$ and $L^{-1}\{g(s)\} = g(t)$ then,

$$L^{-1}\{f(s)\,g(s)\} = \int_0^t f(u)\,g(t-u)\,du$$

Q) Using convolution theorem find $L^{-1}\left\{\frac{1}{s(s-a)}\right\}$

Solution:-

$$L^{-1}\left\{\frac{1}{s}\right\} = g(t) = 1$$

$$L^{-1}\left\{\frac{1}{s-a}\right\} = f(t) = e^{at}$$

$$L^{-1}\left\{\frac{1}{s(s-a)}\right\} = \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t e^{au} \cdot 1 \cdot du = \left.\frac{e^{au}}{a}\right]_0^t = \frac{e^{at}-1}{a}$$

Q) Let $H(s) = \frac{1}{(s-a)(s-b)}$ find $h(t)$?

Soln:-

$$L^{-1}\left\{\frac{1}{s-a}\cdot\frac{1}{s-b}\right\}$$

$$L^{-1}\left\{\frac{1}{s-a}\right\} = f(t) = e^{at}$$

$$L^{-1}\left\{\frac{1}{s-b}\right\} = g(t) = e^{bt}$$

$$\therefore L^{-1}\left\{\frac{1}{s-a}\cdot\frac{1}{s-b}\right\} = \int_0^t f(u)\cdot g(t-u)\, du$$

$$= \int_0^t e^{au}\cdot e^{b(t-u)}\, du = \int_0^t e^{au}\cdot e^{bt-bu}\, du$$

$$= \int_0^t e^{bt}\cdot e^{au-bu}\, du = e^{bt}\int_0^t e^{u(a-b)}\, du$$

$$= e^{bt}\left[\frac{e^{u(a-b)}}{a-b}\right]_0^t$$

$$= \frac{e^{bt}}{a-b}\left[e^{t(a-b)}-1\right]$$

MODULE V

Numerical analysis has a distinct flavour that is different from basic calculus; from solving ODE's algebraically or from other areas. Numeric provides an invaluable extension to the knowledge base of the problem solving engineer.

SOLUTION OF EQUATIONS BY ITERATION

There are two methods for solution of equation by iteration.

i) Newton's method

ii) Fixed point iteration method

ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

$f(x)=0$ is called an algebraic equation if the corresponding $f(x)$ is polynomial. solution of algebraic equation is called roots.

$f(x)=0$ is called transcendental equation if the $f(x)$ is contains trigonometric or exponential or logarithmic equs. functions.

NEWTON'S - RAPHSON METHOD

given a starting value x_0,

compute, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

Q) Apply Newton's method to solve the algebraic eqn $f(x) = x^3+x-1=0$ Correct to 6 decimel places.

a) start with $x_0=1$

b) Start with $x_0=2$

<u>Solution</u>:

$$f(x) = x^3+x-1 = 0$$

$$f'(x) = 3x^2+1$$

By Newton's Ramphson's formula, we're

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

ie $$x_{n+1} = x_n - \frac{x_n^3+x_n-1}{3x_n^2+1}$$

$$= \frac{2x_n^3+1}{3x_n^2+1}, \quad n=0,1,2 \ldots \text{ --- (1)}$$

Start with $x_0 = 1.000$

<u>Step</u> I :- $n=0$, $x_0=1$

$$(1) \Rightarrow x_1 = \frac{2x_0^3+1}{3x_0^2+1} = \frac{2\times1^3+1}{3\times1^2+1} = 0.75$$

<u>Step</u> II :- $n=1$, $x_1 = 0.75$

$$x_2 = \frac{2x_1^3+1}{3x_1^2+1} = \frac{2\times(0.75)^3+1}{3\times(0.75)^3+1} = 0.686$$

<u>Step</u> III :- $n=2$, $x_2=0.686$

$$x_3 = \frac{2x_2^3+1}{3x_2^2+1} = \frac{2\times(0.686)^3+1}{3\times(0.686)^2+1} = 0.682340$$

<u>Step</u> IV :- $n=3$, $x_3=0.6823$

$$x_4 = \frac{2x_3^3+1}{3x_3^2+1} = \frac{2\times(0.6823)^3+1}{3\times(0.6823)^2+1} = 0.682328$$

We accept 0.682328 as an approximate solution of $f(x) = x^3+x-1=0$ Correct to 6 decimal places.

b) Starting with $x_0 = 2.000$

Step 1:

$$x_1 = \frac{2x_0^3+1}{3x_0^2+1} = \frac{2(2)^3+1}{3(2)^2+1} = 1.307692$$

Step 2:- $n=1$ $x_1 = 1.307692$

$$x_2 = \frac{2x_1^3+1}{3x_1^2+1} = \frac{2\times(1.307692)^3+1}{3\times(1.307692)^2+1}$$

$$= 0.892708$$

Step 3:- $n=2$, $x_2 = 0.892708$

$$x_3 = \frac{2x_2^3+1}{3x_2^2+1} = \frac{2(0.892708)^3+1}{3.(0.892708)^2+1}$$

$$= 0.714539$$

Step 4:- $n=3$, $x_3 = 0.714539$

$$x_4 = \frac{2x_3^3+1}{3x_3^2+1} = \frac{2.(0.714539)^3+1}{3.(0.714539)^2+1}$$

$$= \underline{\underline{0.683193}}$$

Q) Apply Newton's method to the following equations, starting from the given x_0 and performing 3 steps.

1) $x^3-5x+3=0$, $x_0 = 2$.

2) $x^4-x^3-2x-34=0$, $x_0 = 3$

3) $x^3-3.9x^2+4.79x-1.881=0$, $x_0=1$

2) Find the positive solution of the transcendental equations $2\sin x = x$, $x_0 = 2$.

Solution:-

$f(x) = 0 \Rightarrow 2\sin x - x = 0$. [Transcendental equations]

By Newton's Raphson's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$f(x_n) = 2\sin x_n - x_n$

$f'(x_n) = 2\cos x_n - 1$

$$x_{n+1} = x_n - \frac{(2\sin x_n - x_n)}{2\cos \cancel{\cos} x_n - 1}$$

$$= \frac{2x_n\cos x_n - x_n - 2\sin x_n + x_n}{2\cos x_n - 1}$$

$$x_{n+1} = \frac{2[x_n\cos x_n - \cancel{x_n}\sin x_n]}{2\cos x_n - 1} \quad -(1)$$

Step 1: $n = 0$, $x_0 = 2$

$$x_1 = 2\left[\frac{2\cos 2 - \sin 2}{2\cos 2 - 1}\right]$$

$= 1.900$

Step 2: $n = 1$, $x_1 = 1.900$

$$x_2 = \frac{2[2\cos 1.900 - \sin 1.900]}{(2\cos 1.900 - 1)}$$

$= 1.8965$

Step 3: $n = 2$, $x_2 = 1.8965$

$x_3 = 1.895$

We accept 1.895 as an approximate solution for $f(x) = x - 2\sin x$.

Q) Find root of the equation $x\sin x + \cos x = 0$, $(x_0 = \pi)$.

solution:-

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x_n) = x_n \sin x_n + \cos x_n$$

$$f'(x_n) = x_n \cos x_n + \sin x_n - \sin x_n$$

$$= x_n \cos x_n$$

by N-R method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(\sin x_n \, x_n + \cos x_n)}{x_n \cos x_n}$$

$$x_{n+1} = \frac{x_n^2 \cos x_n - x_n \sin x_n - \cos x_n}{x_n \cos x_n} \quad \text{---(1)}$$

Step I :-

$n=0$, $x_0 = \pi$

$$x_1 = \frac{\pi^2 \cos\pi - \pi \sin\pi - \cos\pi}{\pi \cos\pi}$$

$$= 2.8233$$

Step II :

$n=1$, $x_1 = 2.8233$

$$x_2 = \frac{(2.8233)^2 \cos(2.8233) - 2.8233 \cdot \sin(2.8233) - \cos(2.8233)}{(2.8233)\,(\cos(2.8233))}$$

$$= \underline{2.7986}$$

Step III :-

$$x_3 = \underline{\underline{2.798}}.$$

Q) Find the real root of the equation $x = e^{-x}$ using N-R method. $(x_0 = 1)$

Solution:-

$f(x) = e^{-x} - x = 0$

$f(x_n) = e^{-x_n} - x_n$

$f'(x_n) = -e^{-x_n} - 1$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$= x_n - \frac{(e^{-x_n} - x_n)}{-e^{-x_n} - 1} = \frac{-x_n e^{-x_n} - x_n - e^{-x_n} + x_n}{-e^{-x_n} - 1}$

$x_{n+1} = \frac{e^{-x_n}[1 + x_n]}{1 + e^{-x_n}}$ —(1)

step I: $n = 0$, $x_0 = 1$

$x_1 = \frac{e^{-1}(1+1)}{1+e^{-1}} = 0.537$

Step II: , $n = 1$

$x_2 = \frac{e^{-0.537}(1+0.537)}{1+e^{-0.537}} = \underline{0.567}$

Step 3:- , $n = 2$, $x_2 = 0.567$

$x_3 = \frac{e^{-0.566}(1+0.567)}{e^{-0.567}+1}$

$= \underline{0.567}$

Q) Set up a Newton iteration formula for computing the square root of a given positive number. Using the same find the square root of 2 exact to six decimal places.

Solution :-

Let c be a given positive number and let x be its positive square root, so that $x = \sqrt{c}$.

Then $x^2 = c$

$x^2 - c = 0$

$f(x) = x^2 - c$

$f(x_n) = x_n^2 - c$

N.R formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_n^2 - c)}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + c}{2x_n} \quad \text{--- (1)}$$

To find square root of 2; $c = 2$.

$$\therefore x_{n+1} = \frac{x_n^2 + 2}{2x_n}$$

Step 1: $x_0 = 1$ (choose)

$$x_1 = \frac{1^2 + 2}{2 \times 1} = 1.500000$$

Step 2:-

$$x_2 = \frac{(1.500000)^2 + 2}{2.(1.500000)} = 1.416667$$

Step 3:-

$x_3 = 1.414216$

Step 4:-

$x_4 = 1.414214$. accept $x_4 = 1.414214$ as the square root of 2.

Q Design Newton iteration for the cube root. Calculate $\sqrt[3]{7}$ starting from $x_0 = 2$ and performing 3 steps.

Solution:-

Let $x = \sqrt[3]{7} = 7^{1/3}$ ie $x = c^{1/3}$ c is any number

ie $x^3 = 7$ $\quad x^3 = c$

$f(x) = x^3 - 7 = 0$ $\quad f(x) = x^3 - c = 0$

$f(x_n) = x_n^3 - c$, $f'(x_n) = 3x_n^2$

N-R method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(x_n^3 - c)}{3x_n^2}$$

$$x_{n+1} = \frac{2x_n^3 + c}{3x_n^2} \quad \text{--- (1)}$$

Step 1:- $n = 0$, $x_0 = 2$, $c = 7$

$$x_1 = \frac{2 \times (2^3) + 7}{3 \cdot (2^2)} ,$$

Step 2: $n = 1$,

$x_2 =$

Step 3: $n = 2$,

$x_3 =$

2) FIXED POINT ITERATION METHOD

To find the solution of the equation $f(x)=0$ —① we transform eqn (1) to the form $x=g(x)$. take an arbitrary value x_0 and then compute x_1, x_2 --- etc from a relation.

$x_{n+1} = g(x_n)$, $n = 0, 1, 2$ ---

solution of $x=g(x)$ is called fixed point of g hence this method is called fixed point iteration method.

Sufficient condition for this iteration process is $|g'(x)| \le k < 1$

Q) Find root of $f(x) = x^2 - 3x + 1 = 0$ by fixed point iteration method.

Solution:-

write the given equation as. $x - 3 + \frac{1}{x} = 0$.

$$x = 3 - \frac{1}{x}.$$

choose $g(x) = 3 - \frac{1}{x}$, then $g'(x) = \frac{1}{x^2}$

$|g'(x)| = \frac{1}{x^2} < 1$ for all $x > 1$.

we can choose $x_0 > 1$

∴ by fixed point iteration method,

$x_{n+1} = g(x_n)$, $n = 0, 1, 2$ ---

$x_{n+1} = 3 - \frac{1}{x_n}$ —(1)

$x_0 = 1.5$, we can choose any number greater than 1.

Step 1:- $n = 0$, $x_0 = 1.5$

(1) ⇒

$x_1 = 3 - \frac{1}{1.5} = 2.3333$

Step 2:-

$x_2 = 3 - \frac{1}{2.3333} = 2.5714$

Step 3 :-

$$x_3 = 3 - \frac{1}{2.5714} = 2.6111$$

Q) Find an approximate root of equation $2x = \cos x + 3$ correct to 3 decimal places.

Start with $x_0 = 1.5$, $x_0 = \frac{\pi}{2}$.

Solution :-

$$2x = \cos x + 3$$

$$x = \frac{\cos x + 3}{2}$$

choose $g(x) = \frac{\cos x + 3}{2}$

$$|g'(x)| = \left|\frac{-\sin x}{2}\right| = \frac{\sin x}{2} < 1 \quad \text{for all } x.$$

Step 1 :-

$n = 0$, $x_0 = 1.5$

$$x_1 = \cos\left(\frac{1.5 + 3}{2}\right) = 1.535$$

Step 2 :- $n = 1$, $x_1 = 1.535$

$$x_2 = \frac{\cos(1.535) + 3}{2} = 1.518$$

Step 3 :-

$$x_3 = \frac{\cos(1.518) + 3}{2} = 1.526$$

Step 4 :-

$$x_4 = \frac{\cos(1.526) + 3}{2} = 1.522$$

b) Start with $x_0 = \pi/2$.

Step 1 : $n = 0$, $x_0 = \pi/2$

$$x_1 = \frac{\cos\left(\frac{\pi}{2}\right) + 3}{2} = 1.5$$

Step 2:- $n=1,\ x_1 = 1.5$

$$x_2 = \frac{\cos(1.5)+3}{2} = 1.535$$

Step 3:- $n=2,\ x_2 = 1.535$

$$x_3 = \frac{\cos(1.535)+3}{2} = 1.518$$

Step 4:- $n=3,\ x_3 = 1.518$

$$x_4 = \frac{\cos(1.518)+3}{4} = 1.527$$

Q) Use the method of iteration find positive root btween 0+1. ($x_0 = 0.5$) of the equation $xe^{x}=1$

Solution:- $x = \frac{1}{e^x} = e^{-x}$

$g(x) = e^{-x}$

$|g'(x)| = e^{-x} < 1$ for all $x < 1$.

$\therefore$ By fixed point iteratuin,

$$x_{n+1} = g(x_n).$$

$$x_{n+1} = e^{-x_n}$$

step 1:- $n=0,\ x_0 = 0.5$

$$x_1 = e^{-0.5} = 0.60653$$

Step 2:- $n=1,\ x_1 = 0.60653$

$$x_2 = e^{-0.60653} = 0.54524$$

step 3:- $n=2,\ x_2 = 0.54524$

$$x_3 = e^{-0.54524} = 0.57970$$

Step 4:- $n=3,\ x_3 = 0.57970$

$$x_4 = e^{-0.57970} = 0.56007$$

LAGRANGE'S INTERPOLATION

Legrange's (n+1) point interpolation formula is given by

$$f(x) \approx L_n(x) = \sum_{k=0}^{n} \frac{I_k(x)}{I_k(x_k)} f_k \quad \longrightarrow (1)$$

where $f_k = f_k(x)$

If we have only two pivotal values $x_0 + x_1$. then equation(1) gives linear Lagrange's interpolation. If we have 3 pivotal values $x_0, x_1 + x_2$, then eqn(1) gives quadratic Legrange's interpolation.

Q) Find Lagrange's interpolation polynomial fitting the points $f(1) = -3$, $f(3) = 0$, $f(4) = 30$, $f(6) = 132$. Hence find $f(5)$?

Solution:

Here 4 tabulated points are given. we need Legrange's polynomial for (n+1) points. $[n+1 = 4 \Rightarrow n = 3]$.

given by, $f(x) \approx L_3(x) = \sum_{k=0}^{3} \frac{I_k(x)}{I_k(x)} f_k$

$$= \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)} \times -3 + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)} \times 0 + \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)} \times 30$$

$$+ \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)} \times 132$$

$$= \frac{(x-3)(x-4)(x-6)}{10} - \frac{(x-1)(x-3)(x-6)}{1} \times 5 + \frac{(x-1)(x-3)(x-4)}{10} \times 44$$

$$= \frac{(x-3)(x-4)(x-6) + 44(x-1)(x-3)(x-4)}{10} - 5(x-1)(x-3)(x-6)$$

$$= \frac{(x-3)(x-4)[x-6+44(x-1)]}{10} - 5(x-1)(x-3)(x-6).$$

$$= \frac{(x-3)(x-4)(45x-50) - 50(x-1)(x-3)(x-6)}{10}$$

Q) The function $y = \sin x$ is tabulated here.

x	$y = \sin x$
0	0
$\pi/4$	0.70711
$\pi/2$	1.0

find $\sin(\pi/6)$?

Solution:-

$n = 2$

$$L_2(x) = \frac{(x-\pi/4)(x-\pi/2)}{(0-\pi/4)(0-\pi/2)} \times 0 + \frac{(x-\pi/2)(x-0) \times .70711}{\left(\frac{\pi}{4}-0\right)\left(\frac{\pi}{4}-\pi/2\right)} + \frac{(x-0)(x-\pi/4) \times 1.0}{\left(\frac{\pi}{2}-0\right)\left(\frac{\pi}{2}-\frac{\pi}{4}\right)}$$

put $x = \frac{\pi}{6}$

$$L_2(x) \approx 0.6285 - 0.111 = 0.51743$$

Q) Certain corresponding values of x + $\log_{10} x$ are $(300, 2.4771)$, $(304, 2.4829)$, $(305, 2.4843)$ + $(307, 2.4871)$. find $\log_{10}(301)$.

Solution:-

$n = 3$

$L_3(x) = \log_{10}(301)$

$$= \frac{(301-304)(301-305)(301-307)}{(300-304)(300-305)(300-307)} 2.4771$$

$$+ \frac{(301-300)(301-305)(301-307)}{(305-300)(305-304)(305-307)} \times 2.4829$$

$$+ \frac{(301-300)(301-304)(301-305)}{(307-300)(307-304)(307-305)} \times 2.4871$$

$$= 2.4786$$

Q) Find

Q) Find Lagrange's interpolation polynomial of degree 2 approximating the function $y = \ln x$ defined by the following table of values. Hence determine the value of $\ln 2.7$?

x :	$y = \ln x$
2 :	0.69315
2.5 :	0.91629
3.0 :	1.09861

Solution:-

here $n = 2$.

$$L_2(x) = \frac{(x-2.5)(x-3)}{(2-2.5)(2-3)} \times 0.69315 + \frac{(x-2)(x-3)}{(2.5-2)(2.5-3)} \times 0.91629 + \frac{(x-2)(x-2.5)}{(3-2)(3-2.5)} \times 1.09861$$

$$= -0.08164x^2 + 0.81366x - 0.60761$$

$$\ln 2.7 = -0.08164 \times 2.7^2 + 0.81366 \times 2.7 - 0.60761$$

$$= \underline{\underline{0.994164}}$$

NEWTON'S FORWARD DIFFERENCE FORMULAE INTERPOLATION FOR EQUAL INTERVALS

Newton's forward difference formula is given by

$$f(x) \approx P_n(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0 + \cdots + \frac{r(r-1)\cdots(r-n-1)}{n!}\Delta^n f_n$$

where $x = x_0 + rh \Rightarrow r = \frac{(x - x_0)}{h}$

Error estimate:

$$\varepsilon_n(x) = f(x) - P_n(x)$$
$$= \frac{h^{n+1} \times r(r-1)\cdots(r-n)}{(n+1)!} f^{n+1}(t)$$

Q) Using Newton's Forward difference interpolation formula and the following table evaluate $f(15)$.

j	x_j	f_j	Δf_j	$\Delta^2 f_j$	$\Delta^3 f_j$	$\Delta^4 f_j$
0	10	46				
			20			
1	20	66		-5		
2	30	81	15		2	-3
			12	-3		
3	40	93			-1	
				-4		
4	50	101	8			

Solution :-

Newton's forward difference interpolation formula is

$$f(x) \approx P_n(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 f_0 \quad —(1)$$

$x = 15$, $f(15) = ?$

$x_0 = 10$, $x_1 = 20$

$h = x_1 - x_0 = 20 - 10 = 10$

$$r = \frac{x - x_0}{h} = \frac{15-10}{10} = \frac{5}{10} = 0.5$$

Substituting all these values in eqn (1).

$f_0 = 46$

$$f(x) \approx f(15) \approx P_4(x) \approx 46 + 0.5 \times 20 + \frac{0.5(0.5-1)}{2!} \times -5 + \frac{0.5(0.5-1)(0.5-2)}{3!} \times 2 + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!} \times -3$$

$$\approx 56.867$$

~~BERNOULLI TRIALS~~

1. There are only two possible outcomes for each trial. [success and failure.]
2. The probability of success is the same for each trial.
3. The outcomes from different trials are independent.

~~BINOMIAL DISTRIBUTION~~

~~Consider~~

Q) Using NFDIF Compute a seven decimal value of the bessels function $J_0(x)$ for $x = 1.72$ from the 4 values in the following table.

J	x_j	$J_0(x_j)$	1^{st} difference	2^{nd} diff.	3^{rd} diff
0	1.7	0.3979849	$^-0.0579685$		
				$^-1.693\times10^{-4}$	
1	1.8	0.3399864	$^-0.0581678$		4.093×10^{-4}
				2.4×10^{-4}	
2	1.9	0.2818186	$^-0.0579278$		
3	2.0	0.2238908			

$x = 1.72$

$x_0 = 1.7$ $\qquad h = x_1 - x_0 = 1.8 - 1.7 = 0.1$

$x_1 = 1.8$

$r = \frac{x - x_0}{h} = \frac{1.72 - 1.7}{0.1} = \underline{\underline{0.2}}$

$f(1.72) = 0.3979849 + (0.2 \cdot {}^-0.0579985) + 0.2\frac{(0.2-1)}{2!}\times {}^-1.693\times10^{-4}$

$+ 0.2\frac{(0.2-1)(0.2-2)}{3!} \times 4.093\times10^{-4} =$

$= 0.3864183$

Q) Using NFDIF, estimate $f(2.05)$ where $f(x) = \sqrt{x}$ using the value,

x :	2.0	2.1	2.2	2.3	2.4
$\sqrt{x}$:	1.414214	1.449138	1.483240	1.516576	1.549193

Solution:-

x	$\sqrt{x}$	Δf_0	$\Delta^2 f_0$	$\Delta^3 f_0$	$\Delta^4 f_0$
2	1.414214				
		0.034924			
2.1	1.449138		-8.22×10^{-4}		
		0.034102		5.5×10^{-5}	
2.2	1.483240		-7.67×10^{-5}		-5×10^{-6}
		0.033335			
2.3	1.516575			5×10^{-5}	
			-7.17×10^{-4}		
2.4	1.549193	0.032618			

$f(2.05) \approx P_4(x)$

$$= f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 f_0$$

$h = x_1 - x_0 = 2.1 - 2 = 0.1$

$r = \frac{x - x_0}{h} = \frac{2.05 - 2}{0.1} = 0.5$

$$\therefore f(2.05) = 1.414214 + 0.5(0.034924) + \frac{0.5(0.5-1)}{2!}\,8.22\times10^{-4} + \frac{0.5(0.5-1)(0.5-2)}{3!}\times 5.5\times10^{-5} + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!}\times 5\times10^{-6}$$

$f(2.05) = 1.431782$

NBDIF:

$$\boxed{f(x) \approx P_n(x) = f_n + r\nabla f_n + \frac{r(r+1)}{2!}\nabla^2 f_n + \cdots + \frac{r(r+1)\cdots(r+n-1)}{n!}\nabla^n f_n \qquad r = \frac{x - x_n}{h}, \; h = x_1 - x_0.}$$

Q) Using NBDI formula Compute seven decimal value of the bessel function $I_0(x)$ for $x = 1.72$ from the 4 values in the following table.

I_{back}	x_j	$I_0(x_j)$	1^{st} difference ∇f	2^{nd} diff. $\nabla^2 f$	3^{rd} diff. $\nabla^3 f$
-3	1.7	0.3979849			
			-0.0579985		
-2	1.8	0.3399864		-1.693×10^{-4}	
			-0.0581678		4.093×10^{-4}
-1	1.9	0.2818186		2.4×10^{-4}	
			-0.0579278		
0	2.0	0.2238908			

Solution:-

Computation of the difference is the same in both cases only the notation differs.

Here last value is taken $= f_n = 0.2238908$.

$x_0 = 1.7$, $x_1 = 1.8$

$h = x_1 - x_0 = 1.8 - 1.7 = 0.1$

$r = \dfrac{x - x_n}{h} = \dfrac{1.72 - 2.0}{0.1} = -2.8$

$\therefore f(1.72) \approx 0.2238908 + (-2.8)(-0.0579278) +$

$\dfrac{(-2.8)(-2.8+1)}{2!} \times 2.4 \times 10^{-4} + \dfrac{(-2.8)(-2.8+1)(-2.8+2)}{3!} \times 4.093 \times 10^{-4}$

$= 0.2238908 + 0.16231 9784 - 6.048 \times 10^{-4} - 2.7504 \times 10^{-4}$

$f(1.72) = \underline{\underline{0.38\ 52086}}$

Sangeetha 06/03/18

GAUSS SEIDAL ITERATION METHOD

Q) Solve the following system of equations using Gauss seidal iteration method in 3 steps starting from 1, 1, 1. $10x+y+z=6$, $x+10y+z=6$, $x+y+10z=6$.

solution:

Given that $x^{(0)}=1$

$y^{(0)}=1$

$z^{(0)}=1$

Now
$$x = 0.6-0.1z-0.1y$$
$$y = 0.6-0.1x-0.1z$$
$$z = 0.6-0.1x-0.1y.$$

Step I:-

using $x^{(0)}=1$, $y^{(0)}=1$, $z^{(0)}=1$, we've

$$x^{(1)} = 0.6-0.1z^{(0)}-0.1y^{(0)} =$$
$$=0.6-0.1\times1-0.1\times1 = 0.4$$

$$y^{(1)} = 0.6-0.1x^{(1)}-0.1z^{(0)}$$
$$=0.6-0.1\times0.4-0.1\times1 = 0.46$$

$$z^{(1)} = 0.6-0.1x^{(1)}-0.1y^{(1)}$$
$$=0.6-0.1\times0.4-0.1\times0.46 = 0.514$$

Step 2:-

$$x^{(2)} = 0.6-0.1z^{(1)}-0.1y^{(1)}$$
$$=0.6-0.1\times0.514-0.1\times0.46 = 0.5026$$

$$y^{(2)} = 0.6-0.1x^{(2)}-0.1z^{(1)}$$
$$=0.6-0.1\times0.5026-0.1\times0.514 = 0.49834$$

$$z^{(2)} = 0.6-0.1x^{(2)}-0.1y^{(2)}$$
$$=0.6-0.1\times0.5026-0.1\times0.49834 = 0.499906$$

Step 3:-

$$x^{(3)} = 0.6 - 0.1z^{(2)} - 0.1y^{(2)}$$

$$= 0.6 - 0.1\times0.499906 - 0.1\times0.49834 = 0.5001754$$

$$y^{(3)} = 0.6 - 0.1x^{(3)} - 0.1z^{(2)}$$

$$= 0.6 - 0.1\times0.5001754 - 0.1\times0.499906 = 0.49999$$

$$z^{(3)} = 0.6 - 0.1x^{(3)} - 0.1y^{(3)}$$

$$= 0.6 - 0.1\times0.5001754 - 0.1\times0.49999 = 0.49998.$$

$\therefore x \approx 0.5, \ y \approx 0.5, \ z \approx 0.5$

Q) Solve using Gauss Seidal method.

$$x_1 - 0.25x_2 - 0.25x_3 = 50$$

$$-0.25x_1 + x_2 - 0.25x_4 = 50$$

$$-0.25x_1 + x_3 - 0.25x_4 = 25$$

$$-0.25x_2 - 0.25x_3 + x_4 = 25$$

$$x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = x_4^{(0)} = 100.$$

Solution:

~~Step~~

$$x_1 = 50 + 0.25x_2 + 0.25x_3$$

$$x_2 = 50 + 0.25x_1 + 0.25x_4$$

$$x_3 = 25 + 0.25x_1 + 0.25x_4$$

$$x_4 = 25 + 0.25x_2 + 0.25x_3$$

Step 1:-

$$x_1^{(1)} = 50 + 0.25x_2^{(0)} + 0.25x_3^{(0)}$$

$$= 50 + 0.25\times100 + 0.25\times100$$

$$= 100$$

$$x_2^{(1)} = 50 + 0.25\,x_1^{(1)} + 0.25\,x_4^{(0)}$$

$$= 50 + 0.25 \times 100 + 0.25 \times 100$$

$$= 100$$

$$x_3^{(1)} = 25 + 0.25\,x_1^{(1)} + 0.25\,x_4^{(0)}$$

$$= 25 + 0.25 \times 100 + 0.25 \times 100$$

$$= 75$$

$$x_4^{(1)} = 25 + 0.25 \times x_2^{(1)} + 0.25\,x_3^{(1)}$$

$$= 25 + 0.25 \times 100 + 0.25 \times 75$$

$$= 68.75$$

Step 2 :-

$$x_1^{(2)} = 50 + 0.25\,x_2^{(1)} + 0.25\,x_3^{(1)}$$

$$= 50 + 0.25 \times 100 + 0.25 \times 75$$

$$= 93.75$$

$$x_2^{(2)} = 50 + 0.25\,x_1^{(2)} + 0.25\,x_4^{(1)}$$

$$= 50 + 0.25 \times 93.75 + 0.25 \times 68.75$$

$$= 90.625$$

$$x_3^{(2)} = 25 + 0.25\,x_1^{(2)} + 0.25\,x_4^{(1)}$$

$$= 25 + 0.25 \times 93.75 + 0.25 \times 68.75$$

$$= 65.625$$

$$x_4^{(2)} = 2.5 + 0.25\,x_2^{(2)} + 0.25\,x_3^{(2)}$$

$$= 2.5 + 0.25 \times 65.625 + 0.25 \times 65.625$$

$$= 64.0625$$

Step 3 :-

$$x_1^{(3)} = 50 + 0.25\,x_2^{(2)} + 0.25\,x_3^{(2)}$$

$$= 50 + 0.25 \times 65.625 + 0.25 \times 65.625$$

$$=$$

$$x_2^{(3)} = 50 + 0.25 x_1^{(2)} + 0.25 x_4^{(2)}$$

$$= 50 + 0.25 \times 93.75 + 0.25 \times 64.0625$$

$$=$$

$$x_3^{(3)} = 25 + 0.25 x_1^{(3)} + 0.25 \times x_4^{(2)}$$

$$= 25 + 0.25 \times \qquad + 0.25 \times 64.0625$$

$$=$$

$$x_4^{(3)} = 25 + 0.25 x_2^{(3)} + 0.25 x_3^{(3)}$$

$$= 25 + 0.25 \times \qquad + 0.25 \times$$

$$=$$

$\therefore x_1 \approx$

$x_2 \approx$

$x_3 \approx$

$x_4 \approx$.

Q)

$$5x_1 - 2x_2 = 18$$

$$-2x_1 + 10x_2 - 2x_3 = -60$$

$$-2x_2 + 15x_3 = 128$$

Do 5 steps starting from [1, 1, 1] using Gauss seidal iteration method.

TRAPEZOIDAL RULE:

In this method we evaluate $\int_a^b f(x)\,dx$ we partition the interval of integration and replace the function $f(x)$ by a straight line segment.

By Trapezoidal rule,

$$J \approx \int_a^b f(x)\,dx = h\left[\frac{1}{2}f(a) + \left(f(x_1) + f(x_2) + \cdots + f(x_{n-1})\right) + \frac{1}{2}f(b)\right].$$

$$= h\left[\frac{1}{2}(f(a)+f(b)) + f(x_1) + \cdots + f(x_{n-1})\right]$$

where $h = \frac{b-a}{n}$, a and b are called nodes.

Q) Evaluate $\int_0^1 e^{-x^2}\,dx$ with $n=10$ using Trapezoidal rule.

Solution:

$$J \approx \int_a^b f(x)\,dx = h\left[\frac{1}{2}[f(a)+f(b)] + [f(x_1) + \cdots + f(x_{n-1})]\right]$$

$$h = \frac{b-a}{n} = \frac{1-0}{10} = \underline{\underline{0.1}}$$

To find the trapezoidal approximation we divide the interval of equation in 10 sub intervals of equal length and use the values in the following table.

J	x_j	x_j^2	$e^{-x_j^2}$
0	0.0	0.0	1.0000
1	0.1	0.01	0.9900498
2	0.2	0.04	0.960789
3	0.3	0.09	0.913931
4	0.4	0.16	0.8521437
5	0.5	0.25	0.778800
6	0.6	0.36	0.6976763
7	0.7	0.49	0.612626
8	0.8	0.64	0.527292
9	0.9	0.81	0.444858
10	1.0	1.0	0.367879

$f(x_1) + \ldots + f(x_{n-1})$

$$J = \int_0^1 e^{-x^2}\,dx$$

$$\approx 0.1\left[\frac{1}{2}(1.00000 + 0.367879) + 6.798459\right]$$

$$\approx \underline{\underline{0.74823}}$$

Q) Use Trapezoidal rule with $n=4$, estimate $\int_1^2 \frac{1}{x}\,dx$.

Solution :-

$$J \approx \int_a^b f(x)\,dx = h\left[\frac{1}{2}(f(a)+f(b)) + f(x_1) + \cdots + f(x_{n-1})\right]$$

$$h = \frac{b-a}{n} = \frac{2-1}{4} = \underline{0.25}$$

J	x_J	$\frac{1}{x_J}$
0	1.00	1
1	1.25	0.8
2	1.50	0.6666
3	1.75	0.57142
4	2.0	0.5

$$J \approx \int_1^2 \frac{1}{x}\,dx$$

$$= 0.25\left[\frac{1}{2}(1+0.5) + 2.03742\right]$$

$$= 0.6968.$$

Q) Use the trapezoidal rule with $n=4$ to estimate $\int_1^2 x^2\,dx$. Compare the estimate with the exact value of the integral.

Solution:-

J	x_J	$f(x_J) = x_J^2$
0	1.00	1.0000
1	1.25	1.5625
2	1.50	2.2500
3	1.75	3.0625
4	2.00	4.0000

$$h = \frac{b-a}{n} = \frac{1}{4} = 0.25$$

$n = 4.$

$$I \approx \int_1^2 x^2\,dx$$

$$= 0.25\left[\frac{1}{2}(1.0000 + 4.0000) + 6.8750\right]$$

$$= 2.34375$$

The exact value of the integral is,

$$\int_1^2 x^2\,dx = \left.\frac{x^3}{3}\right]_1^2 = 2.33334$$

The approximation is a slight over estimate. Each trapezoidal contains slightly more than the corresponding strip under the curve.

Q) Estimate the integral using Trapezoidal rule.

i) $\int_1^2 x\,dx$

ii) $\int_0^2 x^3\,dx$ $\qquad n = 8.$

iii) $\int_{-1}^1 (x^2+1)\,dx.$

SIMPSON'S 1/3 RULE

$$J \approx \int_a^b f(x)\,dx = \frac{h}{3}\left[S_0 + 4S_1 + 2S_2\right]$$

where $S_0 = f_0 + f_{2m}$

$S_1 = f_1 + f_3 + f_5 + \cdots + f_{2m-1}$

$S_2 = f_2 + f_4 + \cdots + f_{2m-2}$

$h = \frac{b-a}{2m}$

Q) Find the approximate value of $\int_0^1 e^{-x^2}dx$ by simpson's 1/3 rule with $2m = 10$.

$J \approx \int_a^b f(x)\,dx = \frac{h}{3}\left[S_0 + 4S_1 + 2S_2\right]$, $h = \frac{b-a}{2m}$

$h = \frac{1-0}{10} = 0.1$

J	x_J	x_J^2	$f_J = e^{-x_J^2}$	
0	0	0.00	1	
1	0.1	0.01	0.99005	
2	0.2	0.04		0.96079
3	0.3	0.09	0.91393	
4	0.4	0.16		0.85214
5	0.5	0.25	0.77880	
6	0.6	0.36		0.69768
7	0.7	0.49	0.61263	
8	0.8	0.64		0.52729
9	0.9	0.81	0.44486	
10	1.0	1.00	0.36787	

$S_0 = f_0 + f_{2m}$
$= 1 + 0.36378 = 1.36378$

$S_1 = f_1 + f_3 + \cdots + f_9$
$= 0.99005 + 0.91393 + 0.77880 + 0.61263 + 0.44486$
$= 3.73935$

$S_2 = 0.96079 + 0.85214 + 0.69768 + 0.52729$
$= 3.0379$

$$J \approx \frac{0.1}{3}\left[1.36378 + 4 \times 3.73935 + 2 \times 3.0379\right]$$

≈ 0.746566

Q) Compute an approximate value of $\int_0^1 x^2\, dx$ by simpson's $\frac{1}{3}$ rule with $2m = 10$.

Solution:-

$$J \approx = \int_a^b f(x)\, dx = \frac{h}{3}\left[S_0 + 4S_1 + 2S_2\right]$$

$S_0 = f_0 + f_{2m}$

$S_1 = f_1 + \cdots + f_{2m-1}$

$S_2 = f_2 + \cdots + f_{2m-2}$

$h = \frac{b-a}{2m} = \frac{1-0}{10} = 0.1$

J	x_j	$f_j = x_j^2$	
0	0.0		0.00
1	0.1	0.01	
2	0.2		0.04
3	0.3	0.09	
4	0.4		0.16
5	0.5	0.25	
6	0.6		0.36
7	0.7	0.49	
8	0.8		0.64
9	0.9	0.81	
10	1.0		1.00

$$J \approx \frac{0.1}{3}\left[(0+1) + 4 \times (1.65) + 2 \times (1.2)\right]$$

$= 0.333$

IMPROVED EULER'S METHOD

$$y_{n+1} = y_n + \frac{1}{2}[k_n + I_n]$$

$$k_n = hf(x_n, y_n)$$

$$I_n = hf(x_{n+1}, y_n + k_n)$$

$x_1 = x_0 + h$

$x_2 = x_1 + h$

$y_0 = y(x_0)$

$y_1 = y(x_1)$

$y_2 = y(x_2)$

Q. Use Improved Euler's method with $h = 0.1$ to find $y(0.2)$ given $\frac{dy}{dx} = x^2 + y^2$ $\quad y(0) = 0$.

Solution:-

$f(x, y) = x^2 + y^2$

$x_0 = 0$, $y_0 = 0$, $h = 0.1$

$x_1 = x_0 + h$

$= 0 + 0.1 = 0.1$

$x_2 = x_1 + h$

$= 0.1 + 0.1 = 0.2$

Using Improved Euler method,

$$y_{n+1} = y_n + \frac{1}{2}[k_n + I_n]$$

$$k_n = h[f(x_n, y_n)]$$

$$I_n = hf(x_{n+1}, y_n + k_n)$$

Using this equation we have to find the values of y_1, y_2, y_3 ---- etc.

Step I:- $n = 0$, $y_0 = 0$

$$y_1 = y_0 + \frac{1}{2}[k_0 + I_0]$$

$$k_0 = hf(x_0, y_0)$$

$$= 0.1 f(0, 0) = 0.1 \times (0^2 + 0^2) = 0$$

$$I_0 = hf(x_1, y_0 + k_0) = 0.1 f(0.1, 0 + 0) = 0.1(0.1^2 + 0^2) = 0.001$$

$y_1 = 0 + \frac{1}{2}(0+0.001) = 0.0005$

Step 2:-

$y_2 = y_1 + \frac{1}{2}(k_1 + I_1)$

$k_1 = hf(x_1, y_1)$

$= 0.1 f(0.1, 0.0005)$

$= 0.1[(0.1)^2 + (0.0005)^2]$

$k_1 = 0.001$

$I_1 = hf(x_2, y_1 + k_1)$

$= 0.1 \times f(0.2, 0.0005 + 0.001)$

$= 0.1 \times f(0.2, 0.006)$

$= 0.1 \times [(0.2)^2 + (0.006)^2]$

$= 0.004$

$y_2 = 0.0005 + \frac{1}{2}[0.001 + 0.004]$

$= 0.003$

$\therefore y(x_2) = y(0.2) = y_2$

$\therefore y(0.2) = 0.003$

Q) Given the IVP $y' = x + y$, $y(0) = 0$, $h = 0.2$. Find the value of y approximately for $x = 1$ by Improved Euler's method in 5 steps.

RUNGE AND KUTTA METHOD

$x_1 = x_0 + h$ $\quad y_1 = y(x_1)$ $\quad y_{n+1} = y_n + \frac{1}{6}[A_n + 2B_n + 2C_n + D_n]$

$x_2 = x_1 + h$ $\quad y_2 = y(x_2)$ $\quad A_n = hf(x_n, y_n)$

$x_3 = x_2 + h$ $\quad y_3 = y(x_3)$ $\quad B_n = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}A_n)$

$\vdots$

$C_n = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}B_n)$

$D_n = hf(x_n + h, y_n + C_n)$

Q) Use Runge kutta method with $h = 0.1$ to find $y(0.2)$ given $\frac{dy}{dx} = x^2 + y^2$ with $y(0) = 0$

Solution:-

$f(x, y) = x^2 + y^2$,

$x_0 = 0$

$y_0 = 0$

$h = 0.1$

Hence $\quad x_1 = x_0 + h = 0 + 0.1 = 0.1$

$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$x_3 = x_2 + h = 0.2 + 0.1 = 0.3$

$y_{n+1} = y_n + \frac{1}{6}[A_n + 2B_n + 2C_n + D_n]$

$A_n = hf(x_n, y_n)$

$B_n = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}A_n)$

$C_n = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}B_n)$

$D_n = hf(x_n + h, y_n + C_n)$

$A_0 = 0.$

Step 1: $n=0, \quad x_0 = 0$

$A_0 = 0.\cancel{2}\, f(x_0\, y_0)$

$= 0.\cancel{1}\, f(0,0) = 0.2\,(0^2+0^2) = 0$ //

$B_0 = 0.\cancel{2}\, f(x_0 + \frac{1}{2}h,\ y_n + \frac{1}{2}A_0)$

$= 0.1 \times f\left(\frac{0.2}{2}, 0\right) = 0.\cancel{2}\,(0.05^2) = 0.00025$

$C_0 = 0.1\, f(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}B_0)$

$= 0.1 \times f\left(0.05, \frac{0.00025}{2}\right) = 0.00025$

$D_0 = 0.1 \times f(x_0 + h,\ 0 + 0.00025)$

$= 0.1\, f(0.\cancel{2}, 0.00025) = 0.001$

$\therefore y_1 = y_0 + \frac{1}{6}(A_0 + 2B_0 + 2C_0 + D_0)$

$= 0 + \frac{1}{6}\left[0 + 2\times 0.00025 + 2\times 0.00025 + \cancel{2}\times 0.001\right]$

$= \underline{\underline{0.00633}}$

Step 2:- $n=1, \quad x_1 = 0.1, \quad x_2 = 0.2$

$A_1 = 0.1\, f(x_1^2 + y_1^2) = 0.1 \times \left[0.1^2 + (0.00033)^2\right] = \underline{0.001}$

$B_1 = 0.1 f(x_1 + \frac{1}{2}h,\ y_1 + \frac{1}{2}A_1)$

$= 0.1\, f\left(0.1 + \frac{0.1}{2},\ 0.00033 + \frac{.001}{2}\right) = 0.00225$

$C_1 = 0.1 f(x_1 + \frac{1}{2}h,\ y_1 + \frac{1}{2}B_1)$

$= 0.0025$

$D_1 = 0.1\, f(x_1 + h,\ y_1 + C_1)$

$= 0.004$

$\therefore y_2 = y_1 + \frac{1}{6}(A_1 + 2B_1 + 2C_1 + D_1) = \underline{\underline{0.002663}}$

$y(0.2) = 0.002663$ //.

Q) Use Runge-Kutta method with $h=0.2$ to find the value of y at $x=0.2$, $x=0.4$ and $x=0.6$ given $\frac{dy}{dx} = 1+y^2$, $y(0)=0$.

Solution:-

Here $f(x,y) = 1+y^2$, $x_0=0$, $y_0=0$, $h=0.2$

Hence $x_1 = x_0+h = 0.2$

$x_2 = x_1+h = 0.4$

To determine y_1, y_2 we use Runge-Kutta method.

$x_{n+1} = x_n + h = x_n + 0.2$

$y_{n+1} = \cancel{hf\ x_n+h}, \ y_n + \frac{1}{6}\left[A_n + 2B_n + 2C_n + D_n\right]$

$A_n = h\,f(x_n, y_n)$

$B_n = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}A_n)$

$C_n = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}B_n)$

$D_n = hf(x_n + h, y_n + C_n)$

step 1: $n=0$

$x_1 = 0.2$

$y_1 = y_0 + \frac{1}{6}\left[A_0 + 2B_0 + 2C_0 + D_0\right]$

$A_0 = 0.2f(0, 0)$

$= 0.2\left[1+0^2\right) = 0.2$

$B_0 = 0.2\left[(1+(y_0 + \frac{1}{2}A_0)^2\right.$

$= 0.2\left[1+(0.1)^2\right] = 0.202$

$C_0 = 0.2\left[1+(y_0 + \frac{1}{2}B_0)^2\right.$

$= 0.2\left[1+(0.101)^2\right] = 0.20204$

$D_0 = 0.2\left[1+(y_0+C_0)^2\right]$

$= 0.2\left[1+(0.20204)^2\right] = 0.20816$

$y_1 = y_0 + \frac{1}{6}(A_0 + 2B_0 + 2C_0 + D_0)$

$= 0 + \frac{1}{6}(0.2 + 2\times 0.202 + 2\times 0.20204 + 0.20816) = 0.2027$

$y(0.2) = \underline{0.2027}$

<u>Step</u> 2:- $n=1$

$x_2 = x_1 + 0.1 = 0.4$

$A_1 = 0.2(1+y_1^2) = 0.2\left[1+(0.2027)^2\right]$

$= 0.2082$

$B_1 = 0.2\left[1+(y_1+\frac{1}{2}A_1)^2\right] = 0.2\left[1+(0.3068)^2\right]$

$= 0.2188$

$C_1 = 0.2\left[1+(y_1+\frac{1}{2}B_1)^2\right] = 0.2\left[1+(0.3121)^2\right]$

$= 0.2195$

$D_1 = 0.2\left[1+(y_1+C_1)^2\right] = 0.2\left[1+(0.4222)^2\right]$

$= 0.2356$

$y_2 = y_1 + \frac{1}{6}\left[A_1 + 2B_1 + 2C_1 + D_1\right]$

$= 0.2027 + \frac{1}{6}\left[0.2082 + 2\times 0.2188 + 2\times 0.2195 + 0.2356\right]$

$= 0.4228$

$y(0.4) = 0.4228$

<u>Step</u> 3:- $n=2$

$x_3 = x_2 + 0.1 = 0.6$

$A_2 = 0.2(1+y_2^2) : 0.2\left(1+(0.4288)^2\right) = 0.237$

$$B_2 = 0.2\left[1+\left(y_2+\frac{1}{2}A_2\right)^2\right]$$

$$= 0.2\left[1+(0.4288+0.1184)^2\right]$$

$$= 0.2598$$

$$D_2 = 0.2\left[1+\left(y_2+C_2\right)^2\right]$$

$$= 0.2\left[1+(0.4288+0.2623)^2\right]$$

$$= 0.295$$

$$C_2 = 0.2\left[1+\left(y_2+\frac{B_2}{2}\right)^2\right]$$

$$= 0.2\left[1+\left(0.4288+\frac{0.2598}{2}\right)^2\right]$$

$$= 0.2623$$

$$\therefore y_3 = y_2 + \frac{1}{6}\left(A_2 + 2B_2 + 2C_2 + D_2\right)$$

$$= 0.4288 + \frac{1}{6}\left(0.237 + 2\times 0.2598 + 2\times 0.2623 + 0.295\right)$$

$$y_3 = 0.6915 = y(0.6)$$